

Derivations on the Algebra of Measurable Operators Affiliated with a Type I von Neumann Algebra

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Abstract

Let M be a type I von Neumann algebra with the center Z , and let $LS(M)$ be the algebra of all locally measurable operators affiliated with M . We prove that every Z -linear derivation on $LS(M)$ is inner. In particular all Z -linear derivations on the algebras of measurable and respectively totally measurable operators are spatial and implemented by elements from $LS(M)$.

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1. Introduction

The present paper is devoted to study of derivations on the algebra of locally measurable operators $LS(M)$ affiliated with a type I von Neumann algebra M .

Given a (complex) algebra \mathcal{A} , a linear operator $D : \mathcal{A} \rightarrow \mathcal{A}$ is called a *derivation* if $D(xy) = D(x)y + xD(y)$ for all $x, y \in \mathcal{A}$. Each element $a \in \mathcal{A}$ generates a derivation $D_a : \mathcal{A} \rightarrow \mathcal{A}$ defined as $D_a(x) = ax - xa$, $x \in \mathcal{A}$. Such derivations are called *inner* derivations.

It is well known that any derivation on a von Neumann algebra is inner and therefore is norm continuous. But the properties of derivations on the unbounded operator algebra $LS(M)$ seem to be very far from being similar. Indeed, the results of [2] and [4] show that in the commutative case when $M = L^\infty(\Omega, \Sigma, \mu)$, where (Ω, Σ, μ) is a non atomic measure space with a finite measure μ , the algebra $LS(M) \cong L^0(\Omega, \Sigma, \mu)$ of all classes of complex measurable functions on (Ω, Σ, μ) admits non zero derivations. It is clear that these derivations are discontinuous in the measure topology (i. e. the topology of convergence in measure), and thus are non inner. In order to avoid such pathological examples we consider derivations on $LS(M)$ which are Z -linear, where Z is the center of the von Neumann algebra M . The main result of the present paper states that if M is a type I von Neumann algebra, then any Z -linear derivation D on the algebra $LS(M)$ is inner, i. e. $D_a(x) = ax - xa$ for an appropriate element $a \in LS(M)$.

In Section 2 we give some preliminaries from the theory of lattice-normed modules and Kaplansky — Hilbert modules over the algebra of measurable functions and recall a result from [1] which gives a description of l -linear derivations on the algebra of all l -bounded l -linear operators on a Banach — Kantorovich space over $l = L^0(\Omega, \Sigma, \mu)$.

In Section 3 we prove that for any type I von Neumann algebra M with the center Z , every Z -linear derivation on the algebra $LS(M)$ of locally measurable operators affiliated with M is inner. As a corollary we obtain that all Z -linear derivations on the algebras of measurable and respectively totally measurable operators affiliated with M are spatial and implemented by elements from $LS(M)$.

2. Preliminaries

Let (Ω, Σ, μ) be a measurable space and suppose that the measure μ has the direct sum property, i. e. there is a family $\{\Omega_i\}_{i \in J} \subset \Sigma$, $0 < \mu(\Omega_i) < \infty$, $i \in J$, such that for any $A \in \Sigma$, $\mu(A) < \infty$, there exist a countable subset $J_0 \subset J$ and a set B with zero measure such that $A = \bigcup_{i \in J_0} (A \cap \Omega_i) \cup B$.

We denote by $l = L^0(\Omega, \Sigma, \mu)$ the algebra of all (classes of) complex measurable functions on (Ω, Σ, μ) equipped with the topology of convergence in measure. Then l is a complete metrizable commutative regular algebra with the unit $\mathbf{1}_\Omega$ given by $\mathbf{1}_\Omega(\omega) = 1, \omega \in \Omega$.

Recall that a net $\{\lambda_\alpha\}$ in L^0 (o)-converges to $\lambda \in L^0$ if there exists a net $\{\xi_\alpha\}$ monotone decreasing to zero such that $|\lambda_\alpha - \lambda| \leq \xi_\alpha$ for all α .

Denote by ∇ the complete Boolean algebra of all idempotents from l , i. e. $\nabla = \{\tilde{\chi}_A : A \in \Sigma\}$, where $\tilde{\chi}_A$ is the element from l which contains the characteristic function of the set A .

A complex linear space E is said to be normed by l if there is a map $\|\cdot\| : E \rightarrow l$ such that for any $x, y \in E, \lambda \in \mathbb{C}$, the following conditions are fulfilled:

- 1) $\|x\| \geq 0; \|x\| = 0 \iff x = 0$;
- 2) $\|\lambda x\| = |\lambda| \|x\|$;
- 3) $\|x + y\| \leq \|x\| + \|y\|$.

The pair $(E, \|\cdot\|)$ is called a lattice-normed space over l . A lattice-normed space E is called d -decomposable, if for any $x \in E$ with $\|x\| = \lambda_1 + \lambda_2, \lambda_i \in l, \lambda_i \geq 0, i = 1, 2, \lambda_1 \lambda_2 = 0$, there exist $x_1, x_2 \in E$ such that $x = x_1 + x_2$ and $\|x_i\| = \lambda_i, i = 1, 2$. A net (x_α) in E is said to be (bo) -convergent to $x \in E$, if the net $\{\|x_\alpha - x\|\}$ (o)-converges to zero in l . A lattice-normed space E which is d -decomposable and complete with respect to the (bo) -convergence is called a *Banach — Kantorovich space*.

It is known that every Banach — Kantorovich space E over l is a module over l and $\|\lambda x\| = |\lambda| \|x\|$ for all $\lambda \in l, x \in E$ (see [5]).

Any Banach — Kantorovich space E over l is orthocomplete, i. e. given any net $(x_\alpha) \subset E$ and a partition of the unit (π_α) in ∇ the series $\sum_\alpha \pi_\alpha x_\alpha$ (bo) -converges in E .

A module F over l is said to be finite-generated, if there exist elements x_1, x_2, \dots, x_n in F such that any $x \in F$ can be decomposed as $x = \lambda_1 x_1 + \dots + \lambda_n x_n$ for appropriate $\lambda_i \in l (i = \overline{1, n})$. The elements x_1, x_2, \dots, x_n are called generators of F . We denote by $d(F)$ the minimal number of generators of F . A module F over l is called σ -finite-generated, if there exists a partition $(\pi_\alpha)_{\alpha \in A}$ of the unit in ∇ such that $\pi_\alpha F$ is finite-generated for any α . A finite-generated module F over l is called homogeneous of type n , if $d(eF) = n$ for every nonzero $e \in \nabla$.

Let \mathcal{K} be a module over l . A map $\langle \cdot, \cdot \rangle : \mathcal{K} \times \mathcal{K} \rightarrow l$ is called an l -valued inner product, if for all $x, y, z \in \mathcal{K}, \lambda \in l$, it satisfies the following conditions:

- 1) $\langle x, x \rangle \geq 0; \langle x, x \rangle = 0 \iff x = 0$;
- 2) $\langle x, y \rangle = \overline{\langle y, x \rangle}$;
- 3) $\langle \lambda x, y \rangle = \lambda \langle x, y \rangle$;
- 4) $\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$.

If $\langle \cdot, \cdot \rangle : \mathcal{K} \times \mathcal{K} \rightarrow l$ is an l -valued inner product, then $\|x\| = \sqrt{\langle x, x \rangle}$ defines an l -valued norm on \mathcal{K} . The pair $(\mathcal{K}, \langle \cdot, \cdot \rangle)$ is called a *Kaplansky — Hilbert module* over l , if $(\mathcal{K}, \|\cdot\|)$ is a Banach — Kantorovich space over l (see [5]).

Let X be a Banach space. A map $s : \Omega \rightarrow X$ is said to be simple, if $s(\omega) = \sum_{k=1}^n \chi_{A_k}(\omega) c_k$, where $A_k \in \Sigma, A_i \cap A_j = \emptyset, i \neq j, c_k \in X, k = \overline{1, n}, n \in \mathbb{N}$. A map $u : \Omega \rightarrow X$ is said to be measurable, if there is a sequence (s_n) of simple maps such that $\|s_n(\omega) - u(\omega)\| \rightarrow 0$ almost everywhere on any $A \in \Sigma$ with $\mu(A) < \infty$.

Let $\mathcal{L}(\Omega, X)$ be the set of all measurable maps from Ω into X , and let $L^0(\Omega, X)$ denote the space of all equivalence classes in $\mathcal{L}(\Omega, X)$ with respect to the equality almost everywhere. Denote by \hat{u} the equivalence class from $L^0(\Omega, X)$ which contains the measurable map $u \in \mathcal{L}(\Omega, X)$. Further we shall identify the element $u \in \mathcal{L}(\Omega, X)$ and the class \hat{u} . Note that the function $\omega \rightarrow \|u(\omega)\|$ is measurable for any $u \in \mathcal{L}(\Omega, X)$. The equivalence class containing the function $\|u(\omega)\|$ is denoted by $\|\hat{u}\|$. For $\hat{u}, \hat{v} \in L^0(\Omega, X), \lambda \in l$ put $\hat{u} + \hat{v} = \widehat{u(\omega) + v(\omega)}, \lambda \hat{u} = \widehat{\lambda(\omega)u(\omega)}$.

It is known [5] that $(L^0(\Omega, X), \|\cdot\|)$ is a Banach — Kantorovich space over l .

Put

$$L^\infty(\Omega) = \{f \in L^0 : \exists c \in \mathbb{R}, c > 0, |f| \leq c\mathbf{1}\}$$

and

$$L^\infty(\Omega, X) = \{x \in L^0(\Omega, X) : \|x\| \in L^\infty(\Omega)\}.$$

Then $L^\infty(\Omega, X)$ is a Banach space with respect to the norm

$$\|x\|_\infty = \|\|x\|\|_{L^\infty(\Omega)}, x \in L^\infty(\Omega, X).$$

If H is a Hilbert space, then $L^0(\Omega, H)$ can be equipped with an l -valued inner product $\langle x, y \rangle = \widehat{(x(\omega), y(\omega))}$, where (\cdot, \cdot) is the inner product on H . Then $(L^0(\Omega, H), \langle \cdot, \cdot \rangle)$ is a Kaplansky — Hilbert module over l and $(L^\infty(\Omega, H), \langle \cdot, \cdot \rangle)$ is a Kaplansky — Hilbert module over $L^\infty(\Omega)$.

Let E be a Banach — Kantorovich space over l . An operator $T : E \rightarrow E$ is l -linear if $T(\lambda_1 x_1 + \lambda_2 x_2) = \lambda_1 T(x_1) + \lambda_2 T(x_2)$ for all $\lambda_1, \lambda_2 \in l, x_1, x_2 \in E$. An l -linear operator $T : E \rightarrow E$ is said to be l -bounded if there exists an element $c \in l$ such that $\|T(x)\| \leq c\|x\|$ for any $x \in E$. For an l -bounded l -linear operator T we put $\|T\| = \sup\{\|T(x)\| : \|x\| \leq \mathbf{1}\}$.

An l -bounded l -linear operator $T : E \rightarrow E$ is called finite-generated (respectively σ -finite-generated) if $T(E) = \{T(x) : x \in E\}$ is a finite-generated (respectively σ -finite-generated) submodule in E .

Denote by $B(E)$ the algebra of all l -bounded l -linear operators on E and let $F_\sigma(E)$ be the set of all σ -finite-generated operators on E .

Let $B(L^\infty(\Omega, H))$ be the set of all $L^\infty(\Omega)$ -bounded $L^\infty(\Omega)$ -linear operators on $L^\infty(\Omega, H)$.

Put

$$B(L^0(\Omega, H))_b = \{x \in B(L^0(\Omega, H)) : \|x\| \in L^\infty(\Omega)\}.$$

Note that the correspondence

$$x \mapsto x|_{L^\infty(\Omega, H)}$$

gives a $*$ -isomorphism between the $*$ -algebras $B(L^0(\Omega, H))_b$ and $B(L^\infty(\Omega, H))$. Further we shall identify $B(L^0(\Omega, H))_b$ with $B(L^\infty(\Omega, H))$ (i. e. the operator x from $B(L^0(\Omega, H))_b$ is identified with its restriction $x|_{L^\infty(\Omega, H)}$).

We shall conclude this section with the following theorem from [1], which is necessary for the proof of the main result of the present paper.

Theorem 2.1. [1]. *Let E be a Banach — Kantorovich space over l and let $D : B(E) \rightarrow B(E)$ be an l -linear derivation. Then there is $T \in B(E)$ such that*

$$D(A) = TA - AT$$

for all $A \in B(E)$.

3. Derivations on the algebra of locally measurable operators for type I von Neumann algebras

Let $B(H)$ be the algebra of all bounded linear operators on a Hilbert space H and let M be a von Neumann algebra in $B(H)$. Denote by $\mathcal{P}(M)$ the lattice of projections in M .

A linear subspace \mathcal{D} in H is said to be affiliated with M (denoted as $\mathcal{D}\eta M$), if $u(\mathcal{D}) \subset \mathcal{D}$ for any unitary operator u from the commutant

$$M' = \{y \in B(H) : xy = yx, \forall x \in M\}$$

of the algebra M .

A linear operator x on H with the domain $\mathcal{D}(x)$ is said to be affiliated with M (denoted as $x\eta M$) if $u(\mathcal{D}(x)) \subset \mathcal{D}(x)$ and $ux(\xi) = xu(\xi)$ for every unitary operator $u \in M'$ and all $\xi \in \mathcal{D}(x)$.

A linear subspace \mathcal{D} in H is said to be strongly dense in H with respect to the von Neumann algebra M , if

- 1) $\mathcal{D}\eta M$;
- 2) there exists a sequence of projections $\{p_n\}_{n=1}^\infty \subset \mathcal{P}(M)$, such that $p_n \uparrow \mathbf{1}$, $p_n(H) \subset \mathcal{D}$ and $p_n^\perp = \mathbf{1} - p_n$ is finite in M for all $n \in \mathbb{N}$, where $\mathbf{1}$ is the identity in M .

A closed linear operator x on a Hilbert space H is said to be *measurable* with respect to the von Neumann algebra M , if $x\eta M$ and $\mathcal{D}(x)$ is strongly dense in H . Denote by $S(M)$ the set of all measurable operators affiliated with M (see [8]).

A closed linear operator x on a Hilbert space H is said to be *locally measurable* with respect to the von Neumann algebra M , if $x\eta M$ and there exists a sequence $\{z_n\}_{n=1}^{\infty}$ of central projections in M such that $z_n \uparrow \mathbf{1}$ and $z_n x \in S(M)$ for all $n \in \mathbb{N}$ (see [10]).

It is known [10] that the set $LS(M)$ of all locally measurable operators affiliated with M forms a unital $*$ -algebra with respect to the strong algebraic operations and the natural involution. Moreover $S(M)$ is a solid $*$ -subalgebra in $LS(M)$.

The following result describes one of the most important properties of the algebra $LS(M)$ (see [6], [7]).

Proposition 3.1. *Suppose that the von Neumann algebra M is the C^* -product of the von Neumann algebras M_i , $i \in I$, where I is an arbitrary set of indices, *i.e.**

$$M = \sum_{i \in I}^{\oplus} M_i = \{ \{x_i\}_{i \in I} : x_i \in M_i, i \in I, \sup_{i \in I} \|x_i\|_{M_i} < \infty \}$$

with coordinate-wise algebraic operations and involution and with the C^* -norm $\|\{x_i\}_{i \in I}\|_M = \sup_{i \in I} \|x_i\|_{M_i}$. Then the algebra $LS(M)$ is $*$ -isomorphic to the algebra $\prod_{i \in I} LS(M_i)$ (with the coordinate-wise operations and involution).

This proposition implies that given any family $\{z_i\}_{i \in I}$ of mutually orthogonal central projections in M with $\bigvee_{i \in I} z_i = \mathbf{1}$ and any family $\{x_i\} \subset LS(M)$ there exists a unique element $x \in LS(M)$ such that $z_i x = z_i x_i$ for all $i \in I$. This element is denoted by $x = \sum_{i \in I} z_i x_i$.

Recall that a von Neumann algebra M is an algebra of *type I* if it is isomorphic to a von Neumann algebra with an abelian commutant.

The main result of the present paper is the following

Theorem 3.2. *Let M be a type I von Neumann algebra with the center Z . Then every Z -linear derivation on the algebra $LS(M)$ is inner.*

The main tool in the proof of this theorem is the decomposition of the given von Neumann algebra into the direct sum of homogeneous components and the representation of homogeneous type I von Neumann algebras as algebras of bounded Z -linear operators on a Kaplansky — Hilbert module over the center Z of the given von Neumann algebra.

For details we refer the reader to the monograph of A.G. Kusraev [5].

First, let us consider the case of a homogeneous type I von Neumann algebra.

Let M be a homogeneous von Neumann algebra of type I_α , where α is a cardinal number, and let $L^\infty(\Omega)$ be the center of M . Then it is known that M is $*$ -isomorphic

to the algebra $B(L^\infty(\Omega, H))$, where $\dim H = \alpha$ (see [3], [5]).

Let (Ω, Σ, μ) be a measure space such that the measure μ has the direct sum property, and let $\{\Omega_i\}_{i \in \mathbb{N}}$ be a countable partition of the set Ω into measurable subsets. Let H_{n_i} denote a finite dimensional Hilbert space with the dimension $n_i, i \in \mathbb{N}$.

Put

$$\sum_{i \in \mathbb{N}}^{\oplus} L^\infty(\Omega_i, H_{n_i}) = \{ \{ \varphi_i \}_{i \in \mathbb{N}} : \varphi_i \in L^\infty(\Omega_i, H_{n_i}), \{ \| \varphi_i \|_i \}_{i \in \mathbb{N}} \in \sum_{i \in \mathbb{N}}^{\oplus} L^\infty(\Omega_i) \},$$

where $\| \cdot \|_i$ is $L^\infty(\Omega_i)$ -valued norm on $L^\infty(\Omega_i, H_{n_i})$. Equipped with coordinate-wise algebraic operations and inner product the set $\sum_{i \in \mathbb{N}}^{\oplus} L^\infty(\Omega_i, H_{n_i})$ becomes a Kaplansky —

Hilbert module over $\sum_{i \in \mathbb{N}}^{\oplus} L^\infty(\Omega_i) \cong L^\infty(\Omega)$ (\cong standing for $*$ -isomorphism).

Similarly, the set

$$\prod_{i \in \mathbb{N}} L^0(\Omega_i, H_{n_i})$$

is a Kaplansky — Hilbert module over $\prod_{i \in \mathbb{N}} L^0(\Omega_i) \cong l$.

Note that $\prod_{i \in \mathbb{N}} L^0(\Omega_i, H_{n_i})$ and $\sum_{i \in \mathbb{N}}^{\oplus} L^\infty(\Omega_i, H_{n_i})$ are the general forms of σ -finite-generated Kaplansky — Hilbert modules over L^0 and over $L^\infty(\Omega)$, respectively.

It should be noted that each finite von Neumann algebra of type I with the center $L^\infty(\Omega)$ is $*$ -isomorphic to an appropriate algebra of the form $B(\sum_{i \in \mathbb{N}}^{\oplus} L^\infty(\Omega_i, H_{n_i}))$, where $\dim H_{n_i} = n_i, n_i \in \mathbb{N}$, and moreover

$$LS(M) = S(M) \cong B(\prod_{i \in \mathbb{N}} (L^0(\Omega_i, H_{n_i}))).$$

Proposition 3.3. *A projection $p \in B(L^\infty(\Omega, H))$ is finite if and only if p is σ -finite-generated.*

Proof. Let $p \in B(L^\infty(\Omega, H))$ be a projection. Since $pB(L^\infty(\Omega, H))p \cong B(p(L^\infty(\Omega, H)))$, p is finite if and only if the algebra $B(p(L^\infty(\Omega, H)))$ is finite. But this exactly means that $p(L^\infty(\Omega, H))$ is σ -finite-generated module, i. e. p is σ -finite-generated projection. The proof is complete.

Denote by $B(L^\infty(\Omega, H)) + F_\sigma(L^0(\Omega, H))$ the $*$ -subalgebra in $B(L^0(\Omega, H))$ which consists of elements of the form $x = x_1 + x_2$, with $x_1 \in B(L^\infty(\Omega, H))$, $x_2 \in F_\sigma(L^0(\Omega, H))$.

Proposition 3.4. *The algebra $S(B(L^\infty(\Omega, H)))$ is $*$ -isomorphic to the algebra $B(L^\infty(\Omega, H)) + F_\sigma(L^0(\Omega, H))$.*

Proof. Take $x \in S(B(L^\infty(\Omega, H)))$ and let $\{e_\lambda\}_{\lambda \in \mathbb{R}}$ be the spectral family of the element $|x|$. By [6, Proposition 1] there exists λ_0 such that $e_{\lambda_0}^\perp$ is a finite projection. We have

$$xe_{\lambda_0}^\perp = e_{\lambda_0}^\perp xe_{\lambda_0}^\perp \in e_{\lambda_0}^\perp S(B(L^\infty(\Omega, H))) e_{\lambda_0}^\perp$$

and

$$e_{\lambda_0}^\perp S(B(L^\infty(\Omega, H)))e_{\lambda_0}^\perp \cong S(e_{\lambda_0}^\perp B(L^\infty(\Omega, H))e_{\lambda_0}^\perp) \cong S(B(e_{\lambda_0}^\perp(L^\infty(\Omega, H)))).$$

By Proposition 3.3 $e_{\lambda_0}^\perp$ is a σ -finite-generated projection and therefore

$$S(B(e_{\lambda_0}^\perp(L^\infty(\Omega, H)))) \cong B(e_{\lambda_0}^\perp(L^0(\Omega, H))).$$

Thus, under the obtained $*$ -isomorphism

$$e_{\lambda_0}^\perp S(B(L^\infty(\Omega, H)))e_{\lambda_0}^\perp \cong B(e_{\lambda_0}^\perp(L^0(\Omega, H)))$$

the element $xe_{\lambda_0}^\perp$ corresponds to some σ -finite-generated operator from $B(e_{\lambda_0}^\perp(L^0(\Omega, H)))$, which is denoted by $\widetilde{xe_{\lambda_0}^\perp}$. Since $xe_{\lambda_0} \in B(L^\infty(\Omega, H))$, we have that the mapping

$$\Phi : x \mapsto xe_{\lambda_0} + \widetilde{xe_{\lambda_0}^\perp}$$

gives a $*$ -embedding of the algebra $S(B(L^\infty(\Omega, H)))$ into $B(L^\infty(\Omega, H)) + F_\sigma(L^0(\Omega, H))$.

Now let $x \in F_\sigma(L^0(\Omega, H))$. Take a σ -finite-generated projection $p \in B(L^\infty(\Omega, H))$ such that $x = pxp$. Then $x = pxp \in pB(L^0(\Omega, H))p$.

Since p is a σ -finite-generated projection, $B(p(L^\infty(\Omega, H)))$ is a finite von Neumann algebra. Hence

$$\begin{aligned} pB(L^0(\Omega, H))p &\cong B(p(L^0(\Omega, H))) \cong S(B(p(L^\infty(\Omega, H)))) \cong \\ &\cong S(pB(L^\infty(\Omega, H))p) \cong pS(B(L^\infty(\Omega, H)))p. \end{aligned}$$

Thus $pB(L^0(\Omega, H))p \cong pS(B(L^\infty(\Omega, H)))p$. Hence, under this $*$ -isomorphism the operator x corresponds to some element from $pS(B(L^\infty(\Omega, H)))p \subset S(B(L^\infty(\Omega, H)))$, and therefore the mapping Φ is surjective.

This means that Φ is a $*$ -isomorphism between $S(B(L^\infty(\Omega, H)))$ and $B(L^\infty(\Omega, H)) + F_\sigma(L^0(\Omega, H))$. The proof is complete.

Proposition 3.5. *The algebras $LS(B(L^\infty(\Omega, H)))$ and $B(L^0(\Omega, H))$ are $*$ -isomorphic.*

Proof. Let us show that the $*$ -isomorphism Φ between $S(B(L^\infty(\Omega, H)))$ and $B(L^\infty(\Omega, H)) + F_\sigma(L^0(\Omega, H))$ can be extended to a $*$ -isomorphism between $LS(B(L^\infty(\Omega, H)))$ and $B(L^0(\Omega, H))$.

Let $x \in LS(B(L^\infty(\Omega, H)))$. Consider a sequence $\{z_n\}$ of central projections such that $z_n \uparrow \mathbf{1}$ and $xz_n \in S(B(L^\infty(\Omega, H)))$ for all $n \in \mathbb{N}$. Put $\pi_1 = z_1$, $\pi_n = z_n \wedge z_{n-1}^\perp$ $n \geq 2$, and

$$\Psi(x) = (bo) - \sum_{n \in \mathbb{N}} \pi_n \Phi(\pi_n x_n).$$

Since $B(L^0(\Omega, H))$ is orthocomplete, Ψ is an imbedding of the algebra $LS(B(L^\infty(\Omega, H)))$ into the algebra $B(L^0(\Omega, H))$.

Let now $y \in B(L^0(\Omega, H))$. Take a sequence $\{z_n\}$ of mutually orthogonal central projections such that $\bigvee_{n \in \mathbb{N}} z_n = \mathbf{1}$ and $\|y\|z_n \in L^\infty(\Omega)$. Then $yz_n \in B(L^\infty(\Omega, H))$. Put $x = \sum_{n \in \mathbb{N}} z_n y$. Then $\Psi(x) = y$ and therefore Ψ is a surjective map. This means that Ψ is a $*$ -isomorphism between $LS(B(L^\infty(\Omega, H)))$ and $B(L^0(\Omega, H))$. The proof is complete.

The assertions of Propositions 3.4 and 3.5 become more clear in the following case of homogeneous type I_∞ von Neumann algebra with the discrete center.

Let M be the C^* -product of a countable family of copies of the von Neumann algebra $B(H)$ with $\dim H = \infty$, i. e.

$$M = \sum_{n \in \mathbb{N}}^{\oplus} B(H).$$

Proposition 3.5 and 3.4 imply that

$$LS(M) \cong \prod_{n \in \mathbb{N}} B(H)$$

and

$$S(M) \cong M + \prod_{n \in \mathbb{N}} F(H),$$

where $F(H)$ is the ideal of finite-dimensional operators from $B(H)$.

Now let us consider general von Neumann algebras of type I.

It is well-known [9] that if M is a type I von Neumann algebra then there is a unique (cardinal-indexed) orthogonal family of projections $(q_\alpha)_{\alpha \in I} \subset \mathcal{P}(M)$ with $\sum_{\alpha \in I} q_\alpha = \mathbf{1}$ such that $q_\alpha M$ is a homogeneous type I_α von Neumann algebra, i. e. $q_\alpha M \cong B(L^\infty(\Omega_\alpha, H_\alpha))$, $\dim H_\alpha = \alpha$, and

$$M \cong \sum_{\alpha \in I}^{\oplus} B(L^\infty(\Omega_\alpha, H_\alpha)).$$

Note that if $L^\infty(\Omega)$ is the center of the von Neumann algebra M then $q_\alpha L^\infty(\Omega) \cong L^\infty(\Omega_\alpha)$ for all $\alpha \in I$.

The product

$$\prod_{\alpha \in I} L^0(\Omega_\alpha, H_\alpha)$$

is a Kaplansky — Hilbert module over L^0 with respect to the coordinate-wise algebraic operations and inner product.

The product

$$\prod_{\alpha \in I} B(L^0(\Omega_\alpha, H_\alpha))$$

with the coordinate-wise algebraic operations and involution forms a $*$ -algebra and moreover

$$\prod_{\alpha \in I} B(L^0(\Omega_\alpha, H_\alpha)) \cong B\left(\prod_{\alpha \in I} L^0(\Omega_\alpha, H_\alpha)\right). \quad (1)$$

Now Propositions 3.1, 3.5 and the isomorphism (1) imply

Proposition 3.6. *For $M \cong \sum_{\alpha \in I}^{\oplus} B(L^\infty(\Omega_\alpha, H_\alpha))$ the algebra $LS(M)$ is $*$ -isomorphic to the algebra $B\left(\prod_{\alpha \in I} L^0(\Omega_\alpha, H_\alpha)\right)$.*

From Proposition 3.6 it follows that if M is a type I von Neumann algebra then given any $x \in LS(M)$ there exists a sequence $\{z_n\}$ of mutually orthogonal central projections with $\bigvee_{n \in \mathbb{N}} z_n = \mathbf{1}$ and $z_n x \in M$ for all $n \in \mathbb{N}$;

Proof of the Theorem 3.2. Let $D : LS(M) \rightarrow LS(M)$ be a Z -linear derivation and $\lambda \in l$. Take a sequence (e_n) of projections in Z such that $e_n \lambda \in L^\infty(\Omega) = Z$ and $e_n \uparrow \mathbf{1}$. Then for any n and $x \in LS(M)$ we have $e_n D(\lambda x) = D(e_n \lambda x) = e_n \lambda D(x)$ and therefore $D(\lambda x) = \lambda D(x)$, i. e. any Z -linear derivation on $LS(M)$ is also l -linear. By Proposition 3.6 and Theorem 2.1 we have D is inner. The proof is complete.

Remark 3.7. The condition on the derivation to be Z -linear is crucial in general. This follows from the examples of non zero derivations on the commutative algebra $L^0(0; 1) \cong LS(L^\infty(0; 1))$ given in [2] (see also [4]). These derivations are not inner, and moreover they are not continuous in the measure topology. Another example of discontinuous (and hence non inner) derivation is the following non commutative generalization of the above one.

Example 3.8. Let δ be any of non zero derivations on $L^0(0; 1)$ constructed in [2]. Consider the von Neumann algebra $M = L^\infty(0; 1) \overline{\otimes} M_n(\mathbb{C})$, which can be identified with the algebra of all $n \times n$ matrices $(f_{i,j})_{i,j=1}^n$ with entries from $L^\infty(0; 1)$. Then the algebra $LS(M)$ is $*$ -isomorphic to the algebra $M_n(L^0(0; 1))$ of all $n \times n$ matrices with entries $f_{i,j}$ from the algebra $L^0(0; 1)$.

Define the mapping $D_\delta : M_n(L^0(0; 1)) \rightarrow M_n(L^0(0; 1))$ by

$$D_\delta((f_{i,j})_{i,j=1}^n) = (\delta(f_{i,j}))_{i,j=1}^n.$$

Then it is easy to check that D_δ is a derivation on $M_n(L^0(0; 1))$ which is not Z -linear (where $Z = L^\infty(0; 1)$), and that D_δ is discontinuous and hence can not be inner.

Now let M be a type I von Neumann algebra and let \mathcal{A} be an arbitrary subalgebra of $LS(M)$ containing M . Consider a derivation $D : \mathcal{A} \rightarrow \mathcal{A}$ and let us show that D can be extended to a derivation \tilde{D} on the whole $LS(M)$.

For an arbitrary element $x \in LS(M)$ take a sequence $\{z_n\}$ of mutually orthogonal central projections with $\bigvee_{n \in \mathbb{N}} z_n = \mathbf{1}$ and $z_n x \in M$ for all $n \in \mathbb{N}$.

Put

$$\tilde{D}(x) = \sum z_n D(z_n x). \quad (2)$$

Since every derivation $D : \mathcal{A} \rightarrow \mathcal{A}$ is identically zero on central projections of M , the equality (2) gives a well-defined Z -linear derivation $\tilde{D} : LS(M) \rightarrow LS(M)$ which coincides with D on \mathcal{A} . By Theorem 3.2 the derivation \tilde{D} is inner and therefore D is a spatial derivation on \mathcal{A} , i. e. there exists an element $a \in LS(M)$ such that

$$D(x) = ax - xa$$

for all $x \in \mathcal{A}$.

Therefore we obtain the following

Theorem 3.9. *Let M be a type I von Neumann algebra with the center Z , and let \mathcal{A} be an arbitrary subalgebra in $LS(M)$ containing M . Then any Z -linear derivation on \mathcal{A} is spatial and implemented by an element of $LS(M)$.*

Now let τ be a faithful normal semi-finite trace on the von Neumann algebra M . Recall that a closed linear operator x is said to be τ -measurable (or totally measurable) with respect to the von Neumann algebra M , if $x\eta M$ and its domain $\mathcal{D}(x)$ is τ -dense in H (i. e. $\mathcal{D}(x)\eta M$ and given any $\varepsilon > 0$ there exists a projection $p \in \mathcal{P}(M)$ such that $p(H) \subset \mathcal{D}(x)$ and $\tau(p^\perp) \leq \varepsilon$).

The set $S(M, \tau)$ of all τ -measurable operators with respect to M is a solid $*$ -subalgebra in $S(M)$ (see [6]). Therefore Theorem 3.9 implies

Corollary 3.10. *Let M be a type I von Neumann algebra with the center Z and let D be a Z -linear derivation on $S(M)$ or on $S(M, \tau)$. Then D is spatial and implemented by an element from $LS(M)$.*

Now let M be a type I von Neumann algebra with the atomic center Z , and let $\{q_i\}_{i \in I}$ be the set of all atoms from Z . Consider a derivation D on $S(M, \tau)$. Since $q_i Z \cong q_i \mathbb{C}$ for all $i \in I$, we have $q_i D(\lambda x) = D(q_i \lambda x) = q_i \lambda D(x)$ for all $i \in I$, $\lambda \in Z$, $x \in S(M, \tau)$. Thus $D(\lambda x) = \lambda D(x)$ for any $\lambda \in Z$. This means that in the case of atomic Z any derivation on $S(M, \tau)$ is automatically Z -linear. From this and from Corollary 3.10 we have the following result, which is a strengthening of a result of Weigt [11].

Corollary 3.11. *If M is a von Neumann algebra with the atomic lattice of projections, then every derivation on the algebra $S(M, \tau)$ is spatial, and in particular is continuous in the measure topology.*

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