

# Solitons via Lie–Bäcklund Transformation for 5D Low–energy String Theory

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## Abstract

We apply a non–linear matrix transformation of Lie–Bäcklund type on a seed soliton configuration in order to obtain a new solitonic solution in the framework of the 5D low–energy effective field theory of the bosonic string. The seed solution represents a stationary axisymmetric two–soliton configuration previously constructed through the inverse scattering method and consists of a *massless* gravitational field coupled to a non–trivial chargeless dilaton and to an axion field endowed with charge. We apply a fully parameterized non–linear matrix transformation of Ehlers type on this massless solution and get a massive rotating axisymmetric gravitational soliton coupled to charged axion and dilaton fields. We discuss on some physical properties of both the initial and the generated solitons and fully clarify the physical effect of the non–linear normalized Ehlers transformation on the seed solution.

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# 1 Introduction

We shall apply a non-linear Lie-Bäcklund transformation in order to construct solitonic solutions in the framework of the 5D low-energy effective field theory of the bosonic string. This theory, when toroidally reduced down to two effective space-time dimensions (in the presence of 3 commuting Killing vectors), allows for the application of the inverse scattering method (ISM), and hence, for the construction of 5D solitonic field configurations of gravitational type [1]–[2]. This method was also modified and implemented for special 4D string models in [3]. These schemes generalize the ISM originally developed by Belinski and Zakharov [4] in the framework of 4D General Relativity for vacuum stationary axisymmetric gravitational fields with two Killing vectors. The ISM was further generalized for the  $D$ -dimensional Einstein theory in the presence of  $D - 2$  Killing vectors [5], and recently applied to obtain the  $N$ -soliton static configuration [6] as well as new axisymmetric stationary solutions [7] in the framework of 5D Einstein theory, Einstein–Maxwell [8] and the Einstein–Maxwell with dilaton systems [9].

In the framework of  $D$ -dimensional low-energy effective string theories the problem becomes more complicated since the chiral representations of these field theories have dimensions greater than two and must satisfy non-trivial group conditions. The chiral matrix which parameterizes the low-energy effective field theory of the heterotic string when reduced to two space-time dimensions possesses the  $SO(d+1, d+n+1)/[SO(d+1) \times SO(d+n+1)]$  symmetry group [10], where  $d$  is the number of compactified dimensions and  $n$  is the number of Abelian vector fields of the theory. This chiral representation involves matrices of dimension  $(2d+n+2)$  which must satisfy orthogonal group conditions. As a consequence, the original scheme of Belinski and Zakharov must be suitably modified and generalized [1]. However, it is not a trivial task to overcome this problem since the group symmetry condition of the chiral matrices and their dimensionality strongly restrict the solitons we can construct, eventually leading to even trivial solutions.

The physical interpretation of the seed soliton we shall use is quite interesting since it describes a massless gravitational object<sup>4</sup> coupled to non-trivial dilaton and Kalb–Ramond fields, which possesses, however, a term which describes angular momentum multiplied by a NUT parameter. Another novel feature of the solitonic seed solution appears when looking towards its rotating black hole/string interpretation; in order to achieve this aim, we must impose the asymptotic flatness condition and set to zero the NUT parameter. However, when one drops the NUT charge, the field configuration becomes static. Thus, rotating solitons necessarily possess a NUT parameter, and, asymptotically flat configurations are necessarily static. In [2], a seed solution of this kind was endowed with gravitational mass and dilaton charge with the aid of a *simplified* non-linear transformation of Lie-Bäcklund type [11], the so-called normalized Ehlers transformation (NET), curing in this way its first strange property. However, the second anomaly remained present in the constructed field configuration since the NET did not affect the physical interpretation of the rotating

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<sup>4</sup>More precisely, an object without mass term in the asymptotical expansion of the  $g_{tt}$ -component.

functions of the metric.

In this paper we apply a fully parameterized NET [12] on the massless seed solution constructed in [2] in order to get a new solitonic object. In this way we can clarify the physical effect of the NET and make an interpretation of the obtained field configuration. These stringy exact solutions are of interest because they can represent new objects that can properly be interpreted in the framework of string theory since in 5D General Relativity there are no uniqueness theorems and different objects, black holes and rings for instance, can be constructed and identified by the same set of parameters.

## 2 Low–energy Effective Action and MEP

In this Section we shall briefly review the field content the 5D low–energy effective field theory of the bosonic string as well as its stationary matrix Ernst potential (MEP) formulation. Then we point out the nonlinear NET symmetry of this stationary theory written in the language of the MEP. Let us start by considering the following action

$$S^{(5)} = \int d^5x |G^{(5)}|^{\frac{1}{2}} e^{-\phi^{(5)}} \left( R^{(5)} + \phi_{;M}^{(5)} \phi^{(5);M} - \frac{1}{12} H_{MNP}^{(5)} H^{(5)MNP} \right), \quad (1)$$

where  $G_{MN}^{(5)}$  is the metric,  $H_{MNP}^{(5)} = \partial_M B_{NP}^{(5)} + \text{cycl. perms. of } M, N, P$ ;  $B_{MN}^{(5)}$  is the anti-symmetric Kalb–Ramond tensor field,  $\phi^{(5)}$  is the dilaton and  $M, N, P = 1, 2, \dots, 5$ . The stationary theory which results after the Kaluza–Klein compactification of this model on  $T^2$  possesses the  $SO(3, 3)/[SO(3) \times SO(3)]$  symmetry group and describes an effective three–dimensional gravity parameterized by the metric tensor

$$g_{\mu\nu} = e^{-2\phi} \left[ G_{\mu,\nu}^{(5)} - G_{p+3,\mu}^{(5)} G_{q+3,\nu}^{(5)} (G^{-1})_{pq} \right] \quad (2)$$

coupled to the following fields: the three–dimensional dilaton field  $\phi$ , and the moduli  $2 \times 2$ –matrices  $G \equiv G_{pq}$  and  $B \equiv B_{pq}$

$$\phi = \phi^{(5)} - \frac{1}{2} \ln |\det G|, \quad G_{pq} = G_{p+3,q+3}^{(5)}, \quad B_{pq} = B_{p+3,q+3}^{(5)}, \quad (3)$$

where  $p, q = 1, 2$  label the time and the fifth coordinate, respectively; the vector fields represented by the  $2 \times 3$ –matrices

$$(A_1)_\mu^p = \frac{1}{2} (G^{-1})_{pq} G_{q+3,\mu}^{(5)}, \quad (A_2)_\mu^{p+2} = \frac{1}{2} B_{p+3,\mu}^{(5)} - B_{pq} A_\mu^q, \quad (4)$$

where  $\mu, \nu = 1, 2, 3$ ; and the antisymmetric tensor field  $B_{\mu\nu}$  (which we set to zero since it does not possess dynamical properties in three dimensions)

$$B_{\mu\nu} = B_{\mu\nu}^{(5)} - 4B_{pq} A_\mu^p A_\nu^q - 2 \left( A_\mu^p A_\nu^{p+2} - A_\nu^p A_\mu^{p+2} \right). \quad (5)$$

In three spatial dimensions, the vector fields can be dualized on-shell through the pseudoscalar fields  $u$  and  $v$  in the following form

$$\begin{aligned}\nabla \times \vec{A}_1 &= \frac{1}{2}e^{2\phi}G^{-1}(\nabla u + B\nabla v), \\ \nabla \times \vec{A}_2 &= \frac{1}{2}e^{2\phi}G\nabla v - B\nabla \times \vec{A}_1,\end{aligned}\tag{6}$$

where all vector and differential operations are performed with respect to the three-dimensional metric  $g_{\mu\nu}$ .

Thus, the effective stationary theory describes three-dimensional gravity  $g_{\mu\nu}$  coupled to the scalar fields  $G$ ,  $B$ ,  $\phi$  and to the pseudoscalar ones  $u$ ,  $v$ .

In [13] it was shown that all these matter fields can be arranged in the following  $3 \times 3$ -matrix Ernst potential

$$\mathcal{X} = \begin{pmatrix} -e^{-2\phi} + v^T X v & v^T X - u^T \\ X v + u & X \end{pmatrix},\tag{7}$$

where  $X$  is a  $2 \times 2$ -matrix potential  $X = G + B$ . The potential  $\mathcal{X}$  was called matrix Ernst potential due to its close analogy to the complex Ernst potential which enters in the stationary formulation of the vacuum Einstein theory [14], which was further generalized for low-energy string theory [15]. The physical meaning of its components is the following: the relevant information concerning the gravitational field is encoded in the matrix potential  $X$  through the moduli matrix  $G$ , whereas its rotational character is parameterized by the dualized variable  $u$ ;  $X$  also contains the antisymmetric Kalb-Ramond tensor field  $B$ , whereas its multidimensional components are dualized through the pseudoscalar field  $v$ , and the 3D dilaton field is represented by  $\phi$ .

In terms of the MEP the effective stationary theory adopts the following form

$${}^3S = \int d^3x |g|^{1/2} \left[ -R + \text{Tr} \left( \frac{1}{4} \nabla \mathcal{X} \mathcal{G}^{-1} \nabla \mathcal{X}^T \mathcal{G}^{-1} \right) \right],\tag{8}$$

where  $\mathcal{G} = (\mathcal{X} + \mathcal{X}^T)/2$  and  $\mathcal{B} = (\mathcal{X} - \mathcal{X}^T)/2$ . The MEP  $\mathcal{X}$  also is the sum of its symmetric and antisymmetric parts:  $\mathcal{X} = \mathcal{G} + \mathcal{B}$  where

$$\mathcal{G} = \begin{pmatrix} -e^{-2\phi} + v^T G v & v^T G \\ G v & G \end{pmatrix}, \quad \mathcal{B} = \begin{pmatrix} 0 & v^T B - u^T \\ B v + u & B \end{pmatrix}.\tag{9}$$

In [12] it was performed a full classification of the symmetries of the stationary action (8) in the language of the MEP according to their charging properties. Among them one encounters a *hidden* symmetry that was called normalized Ehlers transformation. This non-linear symmetry acts in a non-trivial way on the properties of space-time when applying it on a seed solution of the theory, both in the framework of General Relativity and string theory. For instance, by applying the Ehlers transformation on both vacuum Schwarzschild

and Kerr solutions, it generates the so-called NUT charge, changing the topological properties of the space-time [16]. The NET is a matrix generalization of the Lie-Bäcklund symmetry found by Ehlers in the vacuum Einstein theory.

The matrix NET transformation reads

$$\mathcal{X} \rightarrow (1 + \Sigma\Lambda)(1 + \mathcal{X}_0\Lambda)^{-1} \mathcal{X}_0(1 - \Lambda\Sigma) + \Sigma\Lambda\Sigma, \quad (10)$$

where  $\Sigma = \text{diag}(-1, -1; 1)$  and  $\Lambda$  is an arbitrary antisymmetric constant  $3 \times 3$ -matrix  $\Lambda = -\Lambda^T$ . Since this a symmetry of the stationary action (8), it can be used to generate new solutions starting from simple initial solutions. Thus, by applying the NET on a seed solution, we obtain a new stationary solution endowed with three more parameters introduced through the antisymmetric matrix  $\Lambda$ . It is interesting to see what is the physical effect of the NET, and if the introduced parameters can eliminate some of the strange features of the solitonic seed solution we shall consider in the next Section.

### 3 Solution-generating technique

In this Section we shall apply the non-linear NET on a solitonic seed configuration constructed through the ISM in order to analyze the physical effect of the NET and to compare the physical properties of both the initial and the generated solutions.

We first present the explicit expression for the two-soliton seed solution constructed through the ISM in [2]. The 5D metric reads

$$ds_5^2 = G_{pq}(dx^p - \omega_\varphi^p d\varphi)(dx^q - \omega_\varphi^q d\varphi) + e^{2\phi} ds_3^2, \quad (11)$$

where the components of the moduli matrix  $G_{pq}$  are

$$G_{11} = -\frac{r^2 + b^2 - (c - a \cos \theta)^2}{\delta^2}, \quad G_{12} = \frac{2cr}{\delta^2}, \quad G_{22} = \frac{r^2 + b^2 - (c + a \cos \theta)^2}{\delta^2}, \quad (12)$$

the metric functions  $\omega_\varphi^q$  are defined by the relations

$$\omega_\varphi^1 = \frac{-2a\sqrt{bc}r \sin^2 \theta}{\Delta + a^2 \sin^2 \theta}, \quad \omega_\varphi^2 = 2\sqrt{bc} \left[ \cos \theta + \frac{a(b + c - a \cos \theta) \sin^2 \theta}{\Delta + a^2 \sin^2 \theta} \right], \quad (13)$$

the three-dimensional dilaton field has the form

$$e^{2\phi} = 1 - \frac{4bc}{\Delta + a^2 \sin^2 \theta}, \quad (14)$$

the spatial line element is

$$ds_3^2 = (\Delta + a^2 \sin^2 \theta) \left[ \frac{dr^2}{\Delta} + d\theta^2 \right] + \Delta \sin^2 \theta d\varphi^2. \quad (15)$$

The components of the antisymmetric Kalb–Ramond tensor field are

$$B_{12} = \frac{2br}{\delta^2}, \quad B_{4,\varphi}^{(5)} = \frac{2\sqrt{bc}r(2b\cos\theta + a\sin^2\theta)}{\delta^2}, \quad B_{5,\varphi}^{(5)} = 2\sqrt{bc} \left[ \cos\theta - \frac{a(c-b+a\cos\theta)\sin^2\theta}{\delta^2} \right] \quad (16)$$

and the 5D dilaton field reads

$$e^{\phi^{(5)}} = \frac{r^2 + (b-c)^2 - a^2\cos^2\theta}{\delta^2}. \quad (17)$$

Here we have introduced the following notation:  $\sigma^2 = a^2 - (b+c)^2$ ,  $\Delta = r^2 - \sigma^2$  and  $\delta^2 = r^2 + c^2 - (b-a\cos\theta)^2$ , where  $a$ ,  $b$  and  $c$  are constants.

This field configuration represents a stationary axially symmetric *massless* gravitational field coupled to a non-trivial chargeless dilaton field and to an axion field endowed with charge. Despite the massless character of the gravitational field, it possesses angular momentum corresponding to the rotation in the  $\varphi t$ -plane. Namely, by studying the asymptotical behaviour of the metric components  $\omega_\varphi^q$  one sees that  $\omega_\varphi^1$  defines the following angular momentum

$$\omega_\varphi^1 |_{r \rightarrow \infty} \sim \frac{-2a\sqrt{bc}\sin^2\theta}{r}, \quad (18)$$

whereas  $\omega_\varphi^2$  is not asymptotically flat and defines an effective NUT-like parameter:

$$\omega_\varphi^2 |_{r \rightarrow \infty} \sim 2\sqrt{bc}\cos\theta. \quad (19)$$

As pointed out above, as soon as one requires asymptotic flatness, the field configurations become static, whereas the NUT parameter is always present in these stationary solitons. Thus, our family of solutions does not contain a spherically symmetric subclass. It is interesting to note that a comparison to previously found configurations [17] shows that this family of solutions cannot be obtained by setting to zero the mass or other parameters.

### 3.1 New solitonic configuration via NET

In [2] it was proved that the generalization of the ISM proposed by Yurova [1] cannot provide gravitational solitons with mass term. Moreover, in that work, a simplified version of the NET was implemented to cure this strange feature of these ISM solitonic configurations. Here we shall apply the fully parameterized Lie–Bäcklund transformation NET (10) on a seed solution which corresponds to the massless gravitational soliton (11)–(17).

Thus, the seed MEP  $\mathcal{X}_0 = \mathcal{G}_0 + \mathcal{B}_0$  corresponding to this field configuration reads

$$\mathcal{X}_0 = \delta^{-2} \begin{pmatrix} -\delta^2 & 0 & 0 \\ 4\sqrt{bc}(b+c-a\cos\theta) & -[r^2 + b^2 - (c-a\cos\theta)^2] & 2(b+c)r \\ 4\sqrt{bcr} & 2(c-b)r & r^2 + b^2 - (c+a\cos\theta)^2 \end{pmatrix}, \quad (20)$$

where  $\mathcal{G}_0$  and  $\mathcal{B}_0$  are seed matrix potentials that correspond to (9).

Let us make use of the fully parameterized antisymmetric matrix  $\Lambda$ :

$$\Lambda = \begin{pmatrix} 0 & k_1 & k_2 \\ -k_1 & 0 & k_3 \\ -k_2 & -k_3 & 0 \end{pmatrix}, \quad (21)$$

where  $k_1$ ,  $k_2$  and  $k_3$  are arbitrary constants.

After applying the NET on  $\mathcal{X}_0$  with this matrix  $\Lambda$  we obtain a new solution of the theory endowed with three more parameters. Thus, after tedious and lengthy intermediate calculations, the components of the transformed matrix  $G_{pq}$  adopt the form

$$\begin{aligned} G_{11} &= -\frac{K_1(r^2 - a^2 \cos^2 \theta) + 4(K_2 + K_3)r + 2(K_5 + 2K_7)a \cos \theta + (b+c)K_8}{K_1(r^2 - a^2 \cos^2 \theta) + 4(K_2 - K_4)r + 2(K_6 - 2K_7)a \cos \theta - (b+c)K_8}, \\ G_{12} &= \frac{2[(K_5 + 2K_7)r + 2(K_2 + K_3)a \cos \theta]}{K_1(r^2 - a^2 \cos^2 \theta) + 4(K_2 - K_4)r + 2(K_6 - 2K_7)a \cos \theta - (b+c)K_8}, \\ G_{22} &= \frac{K_1(r^2 - a^2 \cos^2 \theta) - 4(K_2 + K_3)r - 2(K_5 + 2K_7)a \cos \theta + (b+c)K_8}{K_1(r^2 - a^2 \cos^2 \theta) + 4(K_2 - K_4)r + 2(K_6 - 2K_7)a \cos \theta - (b+c)K_8}, \end{aligned} \quad (22)$$

where  $K_1 = (1 + k_1^2 - k_2^2 - k_3^2)$ ,  $K_2 = (k_2 + k_1 k_3)\sqrt{bc}$ ,  $K_3 = ck_3 + bk_1 k_2$ ,  $K_4 = bk_3 + ck_1 k_2$ ,  $K_5 = (k_1^2 + k_2^2)b + (1 + k_3^2)c$ ,  $K_6 = (1 + k_3^2)b + (k_1^2 + k_2^2)c$ ,  $K_7 = (k_1 + k_2 k_3)\sqrt{bc}$ , and  $K_8 = (b-c)(1 - k_1^2 + k_2^2 - k_3^2) - 4(k_1 - k_2 k_3)\sqrt{bc}$ ; the metric functions  $\omega_\varphi^q$  read

$$\begin{aligned} \omega_\varphi^1 &= \frac{2K_{10}}{K_1} \cos \theta - \frac{2a[K_9 r + K_{10} a \cos \theta - K_{11}] \sin^2 \theta}{K_1(\Delta + a^2 \sin^2 \theta)}, \\ \omega_\varphi^2 &= \frac{2K_9}{K_1} \cos \theta - \frac{2a[K_{10} r + K_9 a \cos \theta - K_{12}] \sin^2 \theta}{K_1(\Delta + a^2 \sin^2 \theta)}, \end{aligned} \quad (23)$$

where  $K_9 = (k_1 - k_2 k_3)(b-c) + (1 - k_1^2 + k_2^2 - k_3^2)\sqrt{bc}$ ,  $K_{10} = (k_2 - k_1 k_3)(b+c)$ ,  $K_{11} = (b+c)[2(k_3 - k_1 k_2)\sqrt{bc} + (k_2 + k_1 k_3)(b-c)]$ ,  $K_{12} = (b+c)[(1 - k_1^2 - k_2^2 + k_3^2)\sqrt{bc} + (k_1 + k_2 k_3)(b-c)]$ ; the 3D dilaton field is given by the expression

$$e^{2\phi} = 1 + \frac{K_8^2 - (b+c)^2 K_1^2}{K_1^2(\Delta + a^2 \sin^2 \theta)}, \quad (24)$$

which is still chargeless; the components of the antisymmetric Kalb–Ramond field are

$$B_{12} = \frac{2[(K_6 - 2K_7)r + 2(K_2 - K_4)a \cos \theta]}{K_1(r^2 - a^2 \cos^2 \theta) + 4(K_2 - K_4)r + 2(K_6 - 2K_7)a \cos \theta - (b+c)K_8}, \quad (25)$$

and the 5D dilaton field is given by the following relation

$$e^{\phi^{(5)}} = \frac{K_1^2(r^2 - a^2 \cos^2 \theta) + K_8^2}{K_1[K_1(r^2 - a^2 \cos^2 \theta) + 4(K_2 - K_4)r + 2(K_6 - 2K_7)a \cos \theta - (b+c)K_8]}. \quad (26)$$

As a checking exercise, it is straightforward to see that when the  $k_i = 0$  ( $i = 1, 2, 3$ ), one recovers the seed solitonic configuration (11)–(17), and when only  $k_1$  and  $k_2$  vanish, one gets the solution generated in [2].

By studying the asymptotic properties of our generated field configuration one can observe that the gravitational components  $G_{pq}$  have acquired mass terms

$$G_{11} |_{r \rightarrow \infty} \sim -1 + \frac{2m_{11}}{r}, \quad G_{12} |_{r \rightarrow \infty} \sim -\frac{m_{12}}{r}, \quad G_{22} |_{r \rightarrow \infty} \sim 1 - \frac{2m_{22}}{r}, \quad (27)$$

where we have introduced the following effective masses

$$m_{11} = \frac{2(K_3 + K_4)}{(k_2^2 + k_3^2 - 1 - k_1^2)}, \quad m_{12} = \frac{2(2K_7 + K_5)}{(k_2^2 + k_3^2 - 1 - k_1^2)}, \quad m_{22} = \frac{2(K_4 - 2K_2 - K_3)}{(k_2^2 + k_3^2 - 1 - k_1^2)}, \quad (28)$$

whereas the metric functions  $\omega_\varphi^q$  adopt the following asymptotic form

$$\begin{aligned} \omega_\varphi^1 |_{r \rightarrow \infty} &\sim \frac{2K_{10}}{K_1} \cos \theta - \frac{2aK_9 \sin^2 \theta}{K_1 r}, \\ \omega_\varphi^2 |_{r \rightarrow \infty} &\sim \frac{2K_9}{K_1} \cos \theta - \frac{2aK_{10} \sin^2 \theta}{K_1 r}, \end{aligned} \quad (29)$$

making manifest the appearance of an angular momentum term corresponding to the rotation in the  $\varphi x^5$ -plane and an effective NUT charge in the  $\varphi t$ -plane. The transformed component  $B_{12}$  of the antisymmetric tensor field and the 5D dilaton asymptotically behave as

$$B_{12} |_{r \rightarrow \infty} \sim \frac{b_{12}}{r}, \quad e^{\phi^{(5)}} |_{r \rightarrow \infty} \sim 1 + \frac{D}{r}, \quad (30)$$

where we have introduced the following charges

$$b_{12} = \frac{2(2K_7 - K_6)}{(k_2^2 + k_3^2 - 1 - k_1^2)}, \quad D = \frac{4(K_2 - K_4)}{(k_2^2 + k_3^2 - 1 - k_1^2)}. \quad (31)$$

The explicit expressions for the  $B_{3+p,\varphi}^{(5)}$  components of the Kalb–Ramond field are quite involved and we shall omit them, however, we point out that their asymptotic behaviour is very similar to that of the metric functions  $\omega_\varphi^q$ , underlying the torsion nature of this tensor field.

Thus, the solitonic solution we have constructed represents a 5D stationary axisymmetric massive gravitational field configuration coupled to charged dilaton and axion fields. This metric possesses two angular momenta as well as two NUT effective parameters defined in the  $\varphi t$ - and  $\varphi x^5$ - planes. By imposing asymptotic flatness we must set to zero  $K_9$  and  $K_{10}$ , however, in this limit the field configuration becomes static. If just  $K_{10} = 0$  ( $K_9 = 0$ ), we obtain a metric with angular momentum corresponding to rotation in the  $\varphi t$ - ( $\varphi x^5$ -) plane, but with NUT parameter in the  $\varphi x^5$ - ( $\varphi t$ -) plane, as the metric of the initial solution.

## 4 Concluding remarks

In this paper we have constructed a solitonic gravitational field configuration to the 5D low-energy bosonic sector of string theory by applying a non-linear matrix transformation of Lie-Bäcklund type. As a seed solution we used a rotating massless gravitational field configuration coupled to non-trivial dilaton and axion fields which has been obtained previously by means of the ISM. This massless field configuration possesses some strange physical properties. We further endowed this gravitational seed configuration with mass and charge terms, and cured some of its strange physical peculiarities by performing a fully parameterized NET on it.

Thus, the physical meaning of the NET consists of providing masses and charges for the 5D fields as well as generating an angular momentum term and a NUT parameter in the  $\varphi x^5$ - and the  $\varphi t$ -planes, respectively. One more time we see that when we impose the asymptotically flatness condition, we obtain a static field configuration since by setting to zero the effective NUT parameters  $K_9$  and  $K_{10}$ , we force the metric functions (23) to vanish. Thus, if our soliton represents a rotating field configuration, it necessarily possesses NUT parameters and if it constitutes an asymptotically flat solution, it is necessarily static, a feature which is not shared by rotating configurations in General Relativity.

Thus, by comparing the seed and the generated solitonic configurations we see that they share the strange property of being stationary, but with NUT charges, or asymptotically flat, but static. This feature deserves a deeper investigation since both the ISM and the implementation of Lie-Bäcklund transformations could lead to the construction of new charged black hole (brane) solutions in low-energy string theories formulated in  $D > 4$  dimensions as the non-asymptotically flat black hole configurations constructed in [18].

## Acknowledgments

One of the authors (AHA) would like to express his deepest and sincere gratitude to the organizers and all the participants of the International Conference on Geometry, Dynamics and Integrable Systems 2008 for providing a warm and friendly atmosphere during the meeting in Beograd, Serbia. This research was supported by grants CIC-UMSNH-4.16 and CONACYT-60060.

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