Some connections between Falconer's distance set conjecture, and sets of Furstenburg type

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Abstract. In this paper we investigate three unsolved conjectures in geometric combinatorics, namely Falconer's distance set conjecture, the dimension of Furstenburg sets, and Erdös's ring conjecture. We formulate natural δ-discretized versions of these conjectures and show that in a certain sense that these discretized versions are equivalent.

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1. Introduction

In this paper we study Falconer's distance problem, the dimension of sets of Furstenburg type, and Erdös's ring problem. Although we have no direct progress on any of these problems, we are able to reduce the geometric problems to δ-discretized variants and show that these variants are all equivalent.

In order to state the main results we first must develop a certain amount of notation.

1.1. Notation.

$0 < \varepsilon \ll 1$, $0 < \delta \ll 1$ are small parameters. We use $A \lessapprox B$ to denote the estimate $A \leq C_\varepsilon \delta^{-C\varepsilon} B$ for some constants C_ε, C, and $A \approx B$ to denote $A \lessapprox B \lessapprox A$.

We use $\mathbf{B}(x,r) = \mathbf{B}^n(x,r)$ to denote the open ball of radius r centered at x in $\mathbf{R}^n$, and $\mathbf{A} = \mathbf{A}^n$ to denote any annulus in $\mathbf{R}^n$ of the form $\mathbf{A} := \{x : |x| \approx 1\}$.

If A is a finite set, we use $\#A$ to denote the cardinality of A. For finite sets A, B, we say that A is a refinement of B if $A \subset B$ and $\#A \approx \#B$.

If E is contained in a subspace of $\mathbf{R}^n$ and has positive measure in that subspace, we use $|E|$ for the induced Lebesgue measure of E. The subspace will always be clear from context.

For sets E, F of finite measure, we say that E is a refinement of F if $E \subset F$ and $|E| \approx |F|$. We say that E is δ-*discretized* if E is the union of balls of radius $\approx \delta$.

Definition 1.2. For any $0 < \alpha \leq n$, we say that a set E is a $(\delta, \alpha)_n$-set if it is contained in a ball $\mathbf{B}^n(0, C)$, is δ-discretized and one has

$$|E \cap \mathbf{B}(x,r)| \lessapprox \delta^n (r/\delta)^\alpha \tag{1}$$

for all $\delta \leq r \leq 1$ and $x \in \mathbf{R}$.

Roughly speaking, a $(\delta,\alpha)_n$-set behaves like the δ-neighbourhood of an α-dimensional set in $\mathbf{R}^n$. The condition (1) is necessary to ensure that E does not concentrate in a small ball, which would lead to some trivial counterexamples to the conjectures in this paper. (cf. the "two ends" condition in [17], [18]).

If X, Y are subsets of $\mathbf{R}^n$, we use $X+Y$ to denote the set $X+Y := \{x+y : x \in X, y \in Y\}$. Similarly define $X-Y$, and (when $n=1$) $X\cdot Y$, X/Y, X^2, $\sqrt{X}$, etc. Note that $X^2 \subsetneq X \cdot X$ in general. Note that $X \times Y$ denotes the Cartesian product $X \times Y := \{(x,y) : x \in X, y \in Y\}$ as opposed to the pointwise product $X \cdot Y := \{xy : x \in X, y \in Y\}$. Unfortunately there is a conflict of notation between $X^2 := \{x^2 : x \in X\}$ and $X^2 := \{(x,y) : x,y \in X\}$; to separate these two we shall occasionally write the latter as $X^{\oplus 2}$.

If a rectangle R has sides of length a, b for some $a > b$, we call the *direction* of R the direction $\omega \in S^1$ that the sides of length a are oriented on. This is only defined up to sign $\pm$.

1.3. **The Falconer distance problem.** For any compact subset K of the plane $\mathbf{R}^2$, define the *distance set* $\mathrm{dist}(K) \subset \mathbf{R}$ of K by

$$\mathrm{dist}(K) := |K-K| = \{|x-y| : x,y \in K\}.$$

In [8] Falconer conjectured that if $\dim(K) \geq 1$, then $\dim(\mathrm{dist}(K)) = 1$, where $\dim(K)$ denotes the Hausdorff dimension of K. As progress towards this conjecture, it was shown in [8] that $\dim(\mathrm{dist}(K)) = 1$ obtained whenever $\dim(K) \geq 3/2$. This was improved to $\dim(K) \geq 13/9$ by Bourgain [2] and then to $\dim(K) \geq 4/3$ by Wolff [21]. These arguments are based around estimates for L^2 circular means of Fourier transforms of Frostman measures. However, it is unlikely that a purely Fourier-analytic approach will be able to improve upon the 4/3 exponent; for a discussion, see [21].

Now suppose that one only assumes that $\dim(K) \geq 1$. An argument of Mattila [12] shows that $\dim(\mathrm{dist}(K)) \geq \frac{1}{2}$. One may ask whether there is any improvement to this result, in the following sense:

Distance Conjecture 1.4. *There exists an absolute constant $c_0 > 0$ such that* $\dim(\mathrm{dist}(K)) \geq \frac{1}{2} + c_0$ *whenever K is compact and satisfies* $\dim(K) \geq 1$.

This is of course weaker than Falconer's conjecture, but remains open.

One may hope to prove this conjecture by first showing a δ-discretized analogue. As a naive first approximation, we may ask the informal question of whether (for $0 < \delta, \varepsilon \ll 1$) the distance set of a $(\delta,1)_2$ set of measure $\approx \delta$ can be (mostly) contained in a $(\delta, 1/2)_1$ set.

Unfortunately, this problem has an essentially negative answer, as the counterexample

$$\{(x_1,x_2) : x_1 = k\sqrt{\delta} + O(\delta), x_2 = O(\sqrt{\delta}) \text{ for some } k \in \mathbf{Z}, k = O(\delta^{-1/2})\} \tag{2}$$

shows[1] . A substantial portion of the distance set of (2) is contained in the δ-neighbourhood of an arithmetic progression of spacing $\delta^{1/2}$, and this is a $(\delta, \frac{1}{2})_1$ set.

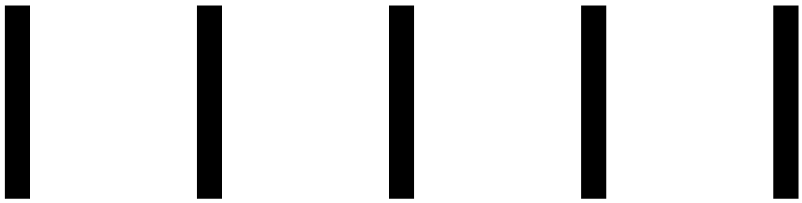

FIGURE 1. An example to remember. Few blurred distances but many blurred points.

This obstruction to solving Conjecture 1.4 can be eliminated by replacing the above informal problem with a "bilinear" variant in which an angular separation condition is assumed:[2]

Bilinear Distance Conjecture 1.5. *Let Q_0, Q_1, Q_2 be three cubes in $\mathbf{B}(0, C)$ of radius ≈ 1 satisfying the separation condition*

$$|(x_1 - x_0) \wedge (x_2 - x_0)| \approx 1 \text{ for all } x_0 \in Q_0, x_1 \in Q_1, x_2 \in Q_2. \tag{3}$$

For each $j = 0, 1, 2$, let E_j be a $(\delta, 1)_2$ subset of Q_j, and let D be a $(\delta, 1/2)_1$ subset of $\mathbf{R}$. Then

$$|\{(x_0, x_1, x_2) \in E_0 \times E_1 \times E_2 : |x_0 - x_1|, |x_0 - x_2| \in D\}| \lessapprox \delta^{3-c_1} \tag{4}$$

where $c_1 > 0$ is an absolute constant.

The estimate (4) is trivially true when $c_1 = 0$. Also, if it were not for condition (3) one could easily disprove (4) for any $c_1 > 0$ by modifying (2). Conjecture 1.5 is also heuristically plausible from analogy with results on the discrete distance problem such as Chung, Szemerédi and Trotter [5]. We remark that the arguments in that paper require the construction of three cubes satisfying (3), and involve the Szemerédi-Trotter theorem (which may be considered as a result concerning the discrete analogue of the Furstenburg problem).

In Section 9 we prove

Theorem 1.6. *A positive answer to the bilinear distance conjecture* 1.5 *implies a positive answer to the distance conjecture* 1.4.

Although this implication looks plausible from discretization heuristics, there are technical difficulties due to the presence of the counter-example (2), and also by the fact that several scales may be in play when studying the Hausdorff dimension of a set.

[1] This counterexample also appears in Fourier-based approaches to the distance problem, see [21]

[2] This idea is frequently used in related problems, see e.g. [15], [1], [20]. Other discretizations are certainly possible, providing of course that (2) is neutralized.

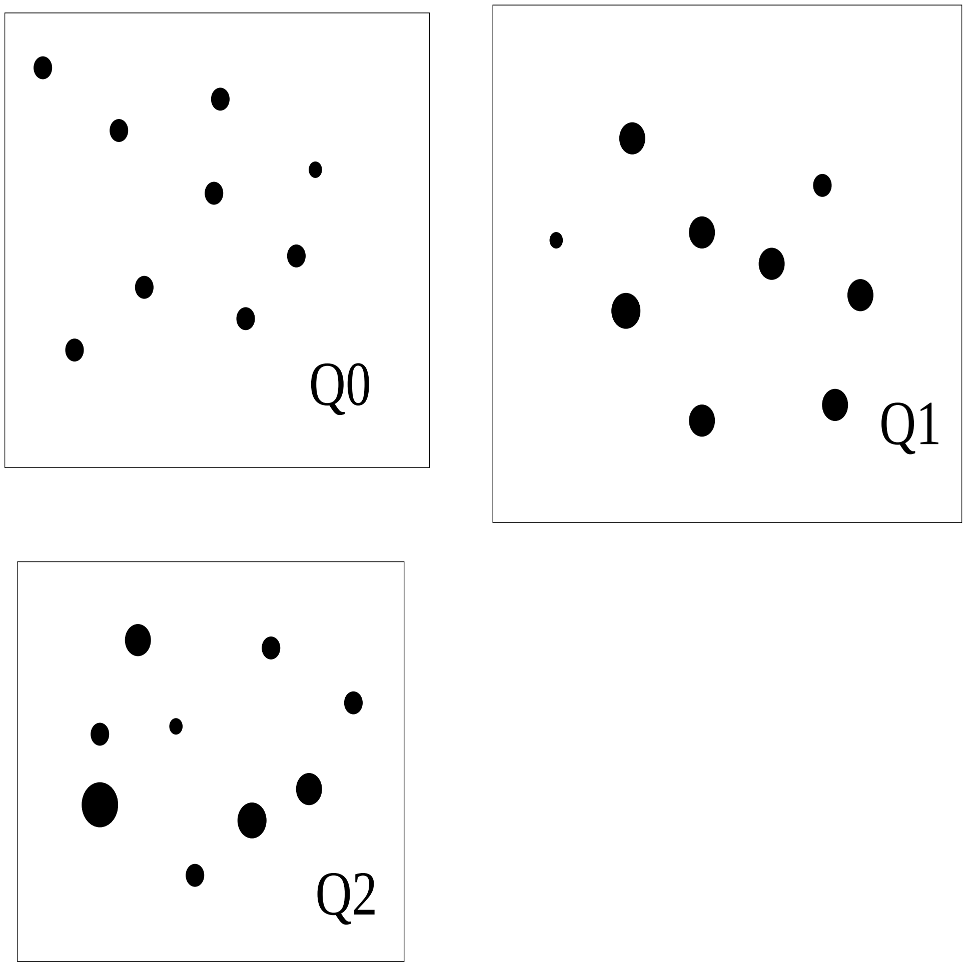

FIGURE 2. In the bilinear distance conjecture, the points are split into three camps.

1.7. **Dimension of sets of Furstenburg type.** We now turn to a problem arising from the work of Furstenburg, as formulated in work of Wolff [19], [21].

Definition 1.8. Let $0 < \beta \leq 1$. We define a β-set to be a compact set $K \subset \mathbf{R}^2$ such that for every direction $\omega \in S^1$ there exists a line segment l_ω with direction ω which intersects K in a set with Hausdorff dimension at least β. We let $\gamma(\beta)$ be the infimum of the Hausdorff dimensions of β-sets.

In [19] the problem of determining $\gamma(\beta)$ is formulated. At present the best bounds known are

$$\max(\beta + \frac{1}{2}, 2\beta) \leq \gamma(\beta) \leq \frac{3}{2}\beta + \frac{1}{2},$$

see [19]. This problem is clearly connected with the Kakeya problem (which is essentially concerned with the higher-dimensional analogue of $\gamma(1)$). Connections to the Falconer distance set problem have also been made, see [21].

The most interesting value of β appears to be $\beta = 1/2$. In this case the two lower bounds on $\gamma(\beta)$ coincide to become $\gamma(\frac{1}{2}) \geq 1$. We ask

Furstenburg problem 1.9. *Is it true that $\gamma(\frac{1}{2}) \geq 1 + c_2$ for some absolute constant $c_2 > 0$? In other words, is it true that $\frac{1}{2}$-sets must have Hausdorff dimension at least $1 + c_2$?*

One can δ-discretize this problem as

Discretized Furstenburg Conjecture 1.10. *Let $0 < \delta \ll 1$, and let Ω be a δ-separated set of directions, and for each $\omega \in \Omega$ let R_ω be a $(\delta, \frac{1}{2})_2$ set contained in a rectangle of dimensions $\approx 1 \times \delta$ oriented in the direction ω. Let E be a $(\delta, 1)_2$ set. Then*

$$|\{(x_0, x_1) \in E \times E : x_1, x_0 \in R_\omega \text{ for some } \omega \in \Omega\}| \lessapprox \delta^{2+c_3} \tag{5}$$

for some absolute constant $c_3 > 0$.

As before, this conjecture is heuristically plausible from analogy with discrete incidence combinatorics, in particular the Szemerédi-Trotter theorem [14]. Unlike the case with the distance problem, the set (2) does not provide a serious threat, and so one does not need to go to a bilinear framework.

The discretized Furstenburg conjecture 1.10 is related to the Furstenburg conjecture 1.9 in much the same way that the Kakeya maximal function conjecture is related to the Kakeya set conjecture. In section 8 we show

Theorem 1.11. *A positive answer to the discretized Furstenburg conjecture* 1.10 *implies a positive answer to the Furstenburg problem* 1.9.

1.12. **The Erdös ring problem.** We consider a problem of Erdös, namely

Ring Problem 1.13. *Does there exist a sub-ring R of* $\mathbf{R}$ *which is a Borel set and has Hausdorff dimension strictly between 0 and 1?*

This problem is connected to Falconer's distance problem; for instance, Falconer [8] used results on the distance problem to show that Borel sub-rings R of $\mathbf{R}$ could not have Hausdorff dimension strictly between 1/2 and 1. Essentially, the idea is to use the fact that $\mathrm{dist}(R \times R) \subseteq \sqrt{R}$.

We concentrate on the specific problem of whether a sub-ring can have dimension exactly 1/2; it seems reasonable to conjecture that such rings do not exist. A positive answer to Conjecture 1.4 would essentially imply this conjecture.

If R is a ring of dimension 1/2, then of course $R + R$ and RR also have dimension 1/2. This leads us to the following δ-discretization of the above conjecture.

Ring Conjecture 1.14. *Let $0 < \delta \ll 1$, and let $A \subset \mathbf{A}$ be a $(\delta, \frac{1}{2})_1$ set of measure $\approx \delta^{1/2}$. Then at least one of $A + A$ and AA has measure $\gtrapprox \delta^{\frac{1}{2}-c_4}$, where $c_4 > 0$ is an absolute constant.*

The dimension condition (1) is crucial, as the trivial counterexample $A := [1, 1 + \delta^{1/2}]$ demonstrates. In principle the discretized ring conjecture gives a negative answer to the Erdös ring problem, but we have not been able to make this rigorous.

For the discrete version of this problem, when measure is replaced by cardinality, there is a result of Elekes [4] that when A has finite cardinality $\#A$, at least one of $A + A$ and AA has cardinality $\gtrapprox \#A^{5/4}$. The proof of this result exploits the Szemerédi-Trotter theorem. This is heuristic evidence for the ring conjecture 1.14 if one accepts the (somewhat questionable) analogy between discrete models and δ-discretized models.

It may appear that the ring hypothesis is being under-exploited when reducing to the ring conjecture 1.14, since one is only using the fact that $R + R$ and RR are small. However, we shall see in Proposition 4.2 that control on $A + A$ and AA actually implies quite good control on other arithmetic expressions such as $AA - AA$ or $(A - A)^2 + (A - A)^2$ (after passing to a refinement), so the ring hypothesis is not being wasted.

One Ring to rule them all,
One Ring to find them,
One Ring to bring them all,
and in the darkness bind them. [16]

1.15. **The main result.** As one can see from the previous discussion, there have been many partial connections drawn between the Falconer, Furstenburg, and Erdös problems. The main result of this paper is to consolidate these connections into

Main Theorem 1.16. *The conjectures* 1.5, 1.10*, and* 1.14 *are logically equivalent.*

We shall prove this theorem in Sections 3-6.

In particular, in order to make progress on the Falconer and Furstenburg problems it suffices to prove the ring conjecture 1.14. This appears to be the easiest of all the above problems to attack. It seems likely that one needs to exploit some sort of "curvature" between addition and multiplication to prove this conjecture, although a naive Fourier-analytic pursuit of this idea seems to run into difficulties. This may indicate that a combinatorial approach will be more fruitful than a Fourier approach. The fact that $\mathbf{R}$ is a totally ordered field may also be relevant, since the analogue of Erdös's ring problem is false for non-ordered fields such as the complex numbers $\mathbf{C}$ or the finite field F_{p^2}. (Unsurprisingly, the analogues of Falconer's distance problem and the conjectures for Furstenburg sets also fail for these fields, see e.g. [19]).

These problems are also related to the Kakeya problem in three dimensions, although the connection here is more tenuous. A proof of Conjecture 1.14 would probably lead (eventually!) to an alternate proof of the main result in [11], namely that Besicovitch sets[3] in $\mathbf{R}^3$ have Minkowski dimension strictly greater than 5/2, and would not rely as heavily on the assumption that the line segments all point in different directions. Very informally, the point is that the arguments in [11] can be pushed a bit further to conclude that a Besicovitch set of dimension exactly 5/2 must essentially be a "Heisenberg group" over a ring of dimension 1/2. We shall not pursue this connection in detail as it is somewhat lengthy and would not directly yield any new progress on the Kakeya problem.

In conclusion, these results indicate that the possibility of 1/2-dimensional rings is a fundamental obstruction to further progress on the Falconer and Furstenburg problems, and may also be obstructing progress on the Kakeya conjecture and related problems (restriction, Bochner-Riesz, Stein's conjecture, local smoothing, etc.) It also appears that substantially new techniques are needed to tackle this obstruction, possibly exploiting the ordering of the reals.

2. Basic tools

In this section $0 < \varepsilon \ll 1$ is fixed, but δ is allowed to vary. As in other sections, the implicit constants here are not allowed to depend on δ.

To clarify many of the arguments in this paper, it may help to know that almost all estimates of the form $A \gtrapprox B$ which occur in this paper are sharp in the sense that the converse bound $A \lessapprox B$ is usually trivial to prove. It is this sharpness which allows us to pass from one expression to another without losing very much in the estimates (if one does not mind the implicit constants in the $\lessapprox$ notation increasing very quickly!).

A typical application of this philosophy is

Cauchy-Schwarz 2.1. *Let A, B be sets of finite measure, and let $\sim$ be a relation between elements of A and elements of B. If*

$$|\{(a,b) \in A \times B : a \sim b\}| \geq \lambda |A||B|$$

for some $0 < \lambda \leq 1$ then

$$|\{(a,b,b') \in A \times B \times B : a \sim b, a \sim b'\}| \geq \lambda^2 |A||B|^2$$

Proof. We can rewrite the hypothesis as

$$\int_A |\{b \in B : a \sim b\}| \; da \geq \lambda |A||B|$$

[3] A Besicovitch set is a set which contains a unit line segment in every direction.

and the conclusion as

$$\int_A |\{b \in B : a \sim b\}|^2 \, da \geq \lambda^2 |A||B|^2.$$

The claim then follows from Cauchy-Schwarz. □

The next lemma deals with the issue of how to refine a δ-discretized set to become a $(\delta, \alpha)_n$ set for suitable α.

Refinement 2.2. *Let $0 < \delta \ll 1$ be a dyadic number, $0 < \alpha < n$, $K \gg 1$ be a constant, and let E be a δ-discretized set in $\mathbf{B}^n(0, C)$ such that $|E| \lessapprox \delta^{n-\alpha}$. Then one can find a set $E_{\delta'}$ for all dyadic $\delta < \delta' \leq 1$ which can be covered by $\lessapprox \delta^{K\varepsilon} \delta'^{-\alpha}$ balls of radius δ', and a set $(\delta, \alpha)_n$ set E^* (with the implicit constants in the definition of a $(\delta, \alpha)_n$ set depending on K) such that*

$$E \subseteq E^* \cup \bigcup_{\delta < \delta' \leq 1} E_{\delta'}.$$

Proof. Define the sets $E_{\delta'}$ by

$$E_{\delta'} := \{x \in \mathbf{R}^n : |E \cap \mathbf{B}(x, \delta')| \geq \delta^{-K\varepsilon} \delta^n (\delta'/\delta)^\alpha\}$$

and E^* by

$$E^* := (E \backslash \bigcup_{\delta < \delta' \leq 1} E_{\delta'}) + \mathbf{B}(0, \delta).$$

The required properties on $E_{\delta'}$ and E^* are then easily verified. □

Separation 2.3. *Let X be a $(\delta, \alpha)_n$ set in $\mathbf{R}^n$ for some $0 < \alpha < n$ such that $|X| \approx \delta^{n-\alpha}$. Then there exist refinements X_1, X_2 of X which respectively live in cubes Q_1, Q_2 of size and separation ≈ 1 with $|Q_1| = |Q_2|$, and $|X_1|, |X_2| \approx \delta^{n-\alpha}$.*

Proof. By (1) we see that

$$|X \cap Q| \leq 10^{-n} |X|$$

for all cubes Q of side-length $\delta^{C_1 \varepsilon}$, if C_1 is a sufficiently large constant. The claim then follows by covering $B(0, C)$ with such cubes, extracting the top 5^n cubes in that collection which maximize $|X \cap Q|$, picking two of those cubes Q_1, Q_2 which are not adjacent, and setting $X_i := X \cap Q_i$ for $i = 1, 2$. We leave the verification of the desired properties to the reader. □

For any function f in $\mathbf{R}^2$, define the Kakeya maximal function $f^*_\delta(\omega)$ for $\omega \in S^1$ by

$$f^*_\delta(\omega) := \sup_R \frac{1}{|R|} \int_R |f|,$$

where R ranges over all $1 \times \delta$ rectangles oriented in the direction ω.

The following estimate can be found in [6] (see also Lemma 6.2):

Kakeya 2.4 (Córdoba's estimate)**.** *We have*

$$\|f_\delta^*\|_2 \lessapprox \|f\|_2.$$

Dually, if we set R_ω *be a collection of* $\delta \times 1$ *rectangles oriented in a* δ*-separated set of directions, then*

$$\|\sum_\omega \chi_{R_\omega}\|_2 \lessapprox 1.$$

3. Arithmetic combinatorics

We shall prove Theorem 1.16 by showing that

$$\text{Bilinear Distance} \implies \text{Discretized Ring} \implies$$
$$\text{Discretized Furstenburg} \implies \text{Bilinear Distance}\ .$$

We shall need a number of standard results concerning the cardinality of sum-sets $A+B$ and difference sets $A-B$, and partial sum-sets $\{a+b : (a,b) \in G\}$, where G is a large subset of $A \times B$.

We first give the results in a discrete setting.

Lemma 3.1. [13] *Suppose* A_1, A_2 *are finite subsets of* $\mathbf{R}$ *such that*

$$\#(A_1 + A_2) \approx \#A_1 \approx \#A_2.$$

Then we have

$$\#(A_{i_1} \pm \ldots \pm A_{i_N}) \approx \#A_1$$

for all choices of signs $\pm$ *and* $i_1, \ldots, i_N \in \{1,2\}$*, where the implicit constants depend on* N*. Also, we can find a refinement* A_1' *of* A_1 *and a real number* x *such that* $x + A_1'$ *is a refinement of* A_2*.*

Proof. Most of these results are in [13]. For the last result, observe that the discrete function $\chi_{-A_1} * \chi_{A_2}$ has an l^1 norm $\approx (\#A_1)^2$ and is supported in a set of cardinality $\approx \#A_1$ by the results in [13]. Thus one can find an x such that $\chi_{-A_1} * \chi_{A_2}(x) \gtrapprox \#A_1$, and the claim follows by setting $A_1' = A_1 \cap (A_2 - x)$. □

We also need Bourgain's variant of the Balog-Szemerédi theorem [3] (as used in Gowers [9]), namely

Lemma 3.2. [3] *Let* $N \gg 1$ *be an integer, and let* A*,* B *be finite subsets of* $\mathbf{R}$ *such that*

$$\#A, \#B \approx N.$$

Suppose there exists a refinement G *of* $A \times B$ *such that*

$$\#\{a+b : (a,b) \in G\} \lessapprox N.$$

Then we can find refinements A'*,* B' *of* A *and* B *respectively such that* $G \cap (A' \times B')$ *is a refinement of* $A' \times B'$*, and for all* $(a', b') \in A' \times B'$ *we have*

$$\#\{(a_1,a_2,a_3,b_1,b_2,b_3) \in A \times A \times A \times B \times B \times B : a' - b' = (a_1 - b_1) - (a_2 - b_2) + (a_3 - b_3)\} \approx N^5.$$

In particular, we have

$$\#(A' - B') \approx N.$$

We can easily replace these discrete lemmata with δ-discretized variants as follows.

Corollary 3.3. *Suppose A, B are finite unions of intervals of length $\approx \delta$ such that*

$$|A + B| \approx |A| \approx |B|.$$

Then we have

$$|A \pm \ldots \pm A| \approx |A|$$

for all choices of signs $\pm$, with the implicit constants depending on the number of signs. Also, we can find a refinement A' of A and a real number x such that $x + A'$ is a refinement of B.

Perfection 3.4. *Let $r \gg \delta$, and let A, B be finite unions of intervals of length $\approx \delta$ such that*

$$|A|, |B| \approx r.$$

Suppose there exists a refinement G of $A \times B$ such that

$$|\{a + b : (a, b) \in G\}| \lessapprox N.$$

Then we can find δ-discretized refinements A', B' of A and B respectively such that $G \cap (A' \times B')$ is a refinement of $A' \times B'$, and for all $(a', b') \in A' \times B'$ we have

$$|\{(a_1, a_2, a_3, b_1, b_2, b_3) \in A \times A \times A \times B \times B \times B : a' - b' = (a_1 - b_1) - (a_2 - b_2) + (a_3 - b_3)\}| \approx r^5.$$

In particular, we have

$$|A' - B'| \approx r.$$

To obtain these corollaries, we first observe that any δ-discretized set A contains the $\approx \delta$-neighbourhood of a discrete set A^* of cardinality $\#A^* \approx |A|/\delta$ which is contained in an arithmetic progression of spacing $\approx \delta$. The claims then follow by applying the previous lemmata to A^*, B^*. (See also the proof of [3], Lemma 2.83).

We also observe the trivial estimate

$$|A + B| \gtrapprox |A|, |B| \tag{6}$$

for all sets A, B.

If we also assume that the sets A, B are contained in the annulus $\mathbf{A}$ then one can also obtain analogues of (6) and the above two Corollaries in which addition and subtraction are replaced by multiplication and division respectively. This simply follows by applying a logarithmic change of variables. In the next section we shall use the fact that multiplication distributes over addition, to obtain hybrid versions of the above results.

4. Bilinear Distance Conjecture 1.5 implies Ring Conjecture 1.14

Assume that the Bilinear Distance Conjecture 1.5 is true for some absolute constant $c_1 > 0$. In this section we show how the Ring Conjecture 1.14 follows.

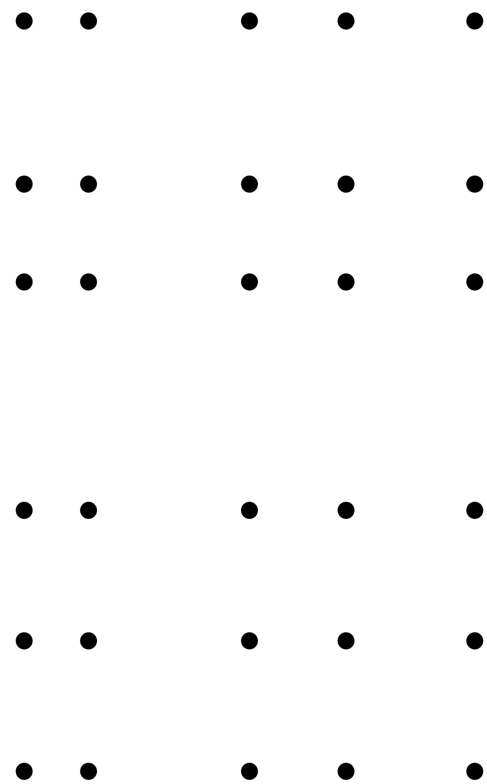

FIGURE 3. A set which contradicts the distance conjecture if a half-dimensional ring exists and constitutes its vertical and horizontal sets of projections.

Let $0 < \varepsilon \ll 1$ be fixed. We may assume that δ is sufficiently small depending on ε, since the Ring Conjecture is trivial otherwise. We may also assume that δ is dyadic. Assume for contradiction that one can find a $(\delta, \frac{1}{2})_1$-set $A \subset \mathbf{A}$ of measure $|A| \approx \delta^{1/2}$ such that

$$|A + A|, |A \cdot A| \lessapprox \delta^{1/2} \tag{7}$$

We will obtain a contradiction from this, and it will be clear from the nature of the argument that one can in fact show that at least one of $A + A$, $A \cdot A$ has measure $\gtrapprox \delta^{\frac{1}{2} - c_4}$ for some absolute constant $c_4 > 0$ depending on c_1.

From Separation 2.3 one can find refinements A_1, A_2 of A which are contained in intervals of size and separation ≈ 1 and have measure $|A_1|, |A_2| \approx \delta^{1/2}$. From the additive and multiplicative versions of (6) we thus have

(8) $$|A_1|, |A_2|, |A_1 + A_2|, |A_1 A_2| \approx \delta^{1/2}.$$

Heuristically, the idea is to apply the Bilinear Distance Conjecture 1.5 with E_0, E_1, E_2 equal to $A_1 \times A_1$, $A_1 \times A_2$, $A_2 \times A_1$ respectively. The difficulty with this is that we cannot quite control the distance set $\sqrt{(A_1 - A_1)^2 + (A_1 - A_2)^2}$ accurately using (8), however this difficulty can be avoided if we pass to various refinements of A.

We turn to the details. From (8) and Perfection 3.4 with A, B, G set to A_1, A_2, $A_1 \times A_2$ respectively, and some re-labeling, we can find δ-discretized refinements C, D of A_1, A_2 respectively such that

(9)
$$|\{(a_1, a_2, a_3, a_4, a_5, a_6) \in A^{\oplus 6} : d - c = (a_1 - a_4) - (a_2 - a_5) + (a_3 - a_6)\}| \approx \delta^{5/2}$$

for all $(c, d) \in C \times D$. From construction we have

(10) $$|c - d| \approx 1 \text{ for all } c \in C, d \in D.$$

Lemma 4.1. *We have*

$$|A{\cdot}A{\cdot}A{\cdot}(C{-}D)/(A{\cdot}A)| = |\{\frac{a_1a_2a_3}{a_4a_5}(c{-}d) : a_1, a_2, a_3, a_4, a_5 \in A, c \in C, d \in D\}| \approx \delta^{1/2}.$$

Proof. The lower bound is clear from (10) and the multiplicative version of (6), so it suffices to show the upper bound.

Fix a_1, a_2, a_3, a_4, a_5, c, d. By multiplying (9) by $a_1a_2a_3/a_4a_5$, which is ≈ 1, we see that

$$|\{(e_1, e_2, e_3, e_4, e_5, e_6) \in (A{\cdot}A{\cdot}A{\cdot}A/(A{\cdot}A))^{\oplus 6} : \frac{a_1a_2a_3}{a_4a_5}(d{-}c) = (e_1{-}e_4){-}(e_2{-}e_5){+}(e_3{-}e_6)\}| \gtrapprox \delta^{5/2}.$$

Integrating this over all possible values of $\frac{a_1a_2a_3}{a_4a_5}(d - c)$ and using Fubini's theorem we obtain

$$|A \cdot A \cdot A \cdot A/(A \cdot A)|^6 \gtrapprox \delta^{5/2} |A \cdot (C - D)|.$$

On the other hand, from (7) and the multiplicative form of Corollary 3.3 we have

$$|A \cdot A \cdot A \cdot A/(A \cdot A)| \approx \delta^{1/2}.$$

The claim follows by combining the above two estimates. □

From (8) and the multiplicative version of (6) we have

$$|C|, |D|, |CD| \approx \delta^{1/2}.$$

From the multiplicative form of Perfection 3.4 with $A := C$ and $B := 1/D$, we may thus find refinements C', D' of C, D respectively such that

(11)
$$|\{(c_1, c_2, c_3, d_1, d_2, d_3) \in C \times C \times C \times D \times D \times D : cd = (c_1d_1)(c_2d_2)^{-1}(c_3d_3)\}| \approx \delta^{5/2}$$

for all $c \in C'$, $d \in D'$.

Lemma 4.2. *We have*

$$|C'D' - C'D'| = |\{cd - c'd' : c, c' \in C', d, d' \in D'\}| \approx \delta^{1/2}.$$

Proof. As before, the lower bound is immediate from the additive and multiplicative versions of (6), so it suffices to show the upper bound.

Fix c, c', d, d', and let X denote the set in (11). Observe that for all $(c_1, c_2, c_3, d_1, d_2, d_3) \in X$ we have the telescoping identity

$$cd - c'd' = x_1 - x_2 + x_3 - x_4$$

where

$$\begin{aligned} x_1 &:= \frac{(c_1 - d')d_1 c_3 d_3}{c_2 d_2} \\ x_2 &:= \frac{d'(c' - d_1)c_3 d_3}{c_2 d_2} \\ x_3 &:= \frac{d'c'(c_3 - d_2)d_3}{c_2 d_2} \\ x_4 &:= \frac{d'c'd_2(c_2 - d_3)}{c_2 d_2}. \end{aligned}$$

Indeed, we have the identities

$$\begin{aligned} \frac{c_1 d_1 c_3 d_3}{c_2 d_2} &= cd \\ \frac{d' d_1 c_3 d_3}{c_2 d_2} &= cd - x_1 \\ \frac{d'c' c_3 d_3}{c_2 d_2} &= cd - x_1 + x_2 \\ \frac{d'c' d_2 d_3}{c_2 d_2} &= cd - x_1 + x_2 - x_3 \\ c'd' = \frac{d'c' d_2 c_2}{c_2 d_2} &= cd - x_1 + x_2 - x_3 + x_4. \end{aligned}$$

As a consequence of these identities, (10) and some algebra we see the

$$(c_1, c_2, c_3, d_1, d_2, d_3) \mapsto (x_1, x_2, x_3, x_4, c_2, d_2)$$

is a diffeomorphism on X (recall that c, d, c', d' are fixed). From (11) we thus have

$$|\{(x_1, x_2, x_3, x_4, c_2, d_2) \in (A{\cdot}A{\cdot}A{\cdot}(C{-}D)/(A{\cdot}A))^{\oplus 4} \times C \times D : cd - c'd' = x_1 - x_2 + x_3 - x_4\}| \gtrapprox \delta^{5/2}.$$

Integrating this over all values of $cd - c'd'$ and using Fubini's theorem we obtain

$$|C'D' - C'D'| \gtrapprox \delta^{5/2} |A \cdot A \cdot A \cdot (C - D)/(A \cdot A)|^4 |C||D|.$$

The claim then follows from Lemma 4.1. □

From the above lemma and the multiplicative form of (6) we have

$$|C'|, |D'|, |C'D'| \approx \delta^{1/2}.$$

From the multiplicative version of Corollary 3.3 we can therefore find a refinement F of C' and a real number $x \approx 1$ such that xF is a refinement of D'. In particular, since $FF - FF$ is a subset of $x^{-1}(C'D' - C'D')$, we thus see that

$$|FF - FF| \approx |FF| \approx |F| \approx \delta^{1/2}.$$

From Corollary 3.3 we thus have

$$|FF - FF - FF + FF + FF - FF - FF + FF| \approx \delta^{1/2}.$$

Since $(F - F)^2 \subset FF - FF - FF + FF$, we thus have

$$|(F - F)^2 + (F - F)^2| \lessapprox \delta^{1/2}.$$

The set F is a $(\delta, \frac{1}{2})_1$ set with measure $\approx \delta^{1/2}$. From Separation 2.3 we may find refinements F_1, F_2 of F which are contained in intervals I_1, I_2 of size and separation ≈ 1 such that $|I_1| = |I_2|$ and $|F_1|, |F_2| \approx \delta^{1/2}$.

Define

$$E_0 := F_1 \times F_1, E_1 := F_1 \times F_2, E_2 := F_2 \times F_1, \quad Q_0 := I_1 \times I_1, Q_1 := I_1 \times I_2, Q_2 := I_2 \times I_1.$$

It is clear that Q_0, Q_1, Q_2 obey (3) and that E_0, E_1, E_2 are $(\delta, 1)_2$ sets of measure $\approx \delta$ contained in Q_0, Q_1, Q_2 respectively.

Let D denote the set

$$D = \sqrt{(F_2 - F_1)^2 + (F_1 - F_1)^2}.$$

Clearly D is a δ-discretized set of measure $|D| \lessapprox \delta^{1/2}$ which lives in $\mathbf{A}$. In fact, from the size and separation of F_1 and F_2 we have

(12) $$|D| \approx \delta^{1/2}.$$

Also, we have

$$|x_1 - x_0|, |x_2 - x_0| \in D$$

for all $x_0 \in E_0$, $x_1 \in E_1$, $x_2 \in E_2$. In particular, we have

(13)
$$|\{(x_0, x_1, x_2) \in E_0 \times E_1 \times E_2 : |x_0 - x_1|, |x_0 - x_2| \in D\}| = |E_0||E_1||E_2| \approx \delta^3.$$

We are almost ready to apply the hypothesis (4), however the one thing which is missing is that D need not satisfy (1). To rectify this we shall remove some portions from D.

Apply Refinement 2.2 to obtain a covering

$$D \subset D^* \cup \bigcup_{\delta < \delta' \ll 1} D_{\delta'}$$

with the properties asserted in Refinement 2.2 , and K equal to a large constant to be chosen shortly.

Proposition 4.3. *For all $\delta' > \delta$, we have*

$$|\{(x_0, x_1) \in E_0 \times E_1 : |x_0 - x_1| \in D_{\delta'}\}| \lessapprox \delta^2 \delta^{K\varepsilon/100}$$

and

$$|\{(x_0, x_2) \in E_0 \times E_2 : |x_0 - x_2| \in D_{\delta'}\}| \lessapprox \delta^2 \delta^{K\varepsilon/100}.$$

Proof. Fix δ'. We may assume that ε is sufficiently small depending on K, and δ is sufficiently small depending on K and ε, since the claim is trivial otherwise.

By reflection symmetry it suffices to prove the first estimate. Suppose for contradiction that

$$|\{(x_0, x_1) \in E_0 \times E_1 : |x_0 - x_1| \in D_{\delta'}\}| \gtrapprox \delta^2 \delta^{K\varepsilon/100}.$$

From Cauchy-Scwartz 2.1 we thus have

$$|\{(x_0, x_1, x_1') \in E_0 \times E_1 \times E_1 : |x_0 - x_1| \in D_{\delta'}, |x_0 - x_1'| \in D_{\delta'}\}| \gtrapprox \delta^3 \delta^{K\varepsilon/50}.$$

Write $x_1 = (x_1, y_1)$, $x_1' = (x_1', y_1')$. Observe that

$$|\{(x_0, x_1, x_1') \in E_0 \times E_1 \times E_1 : |x_1 - x_1'| \lessapprox \delta^{K\varepsilon/10}\}| \lessapprox \delta^3 \delta^{K\varepsilon/20}.$$

This is because for fixed x_1, x_1' can only range in a set of measure $\lessapprox \delta^{1/2}\delta^{K\varepsilon/20}$ thanks to (1) and the fact that F_1 is a $(\delta, \frac{1}{2})_1$ set. Subtracting the two inequalities we obtain (if δ is sufficiently small)

$$|\{(x_0, x_1, x_1') \in E_0 \times E_1 \times E_1 : |x_0 - x_1| \in D_{\delta'}, |x_0 - x_1'| \in D_{\delta'}, |x_1 - x_1'| \gtrapprox \delta^{K\varepsilon/10}\}| \gtrapprox \delta^3 \delta^{K\varepsilon/50}.$$

Since $|E_1| \approx \delta$, we may thus find $x_1, x_1' \in E_1$ such that

$$|x_1 - x_1'| \gtrapprox \delta^{K\varepsilon/10} \tag{14}$$

and

$$|\{x_0 \in E_0 : |x_0 - x_1| \in D_k, |x_0 - x_1'| \in D_{\delta'}\}| \gtrapprox \delta \delta^{K\varepsilon/50}. \tag{15}$$

From Refinement 2.2 $D_{\delta'}$ can be covered by $\lessapprox \delta^{K\varepsilon} \delta'^{-1/2}$ intervals in $\mathbf{A}$ of length $\lessapprox \delta'$. From this fact, (14), and the geometry of annuli which intersect non-tangentially, we see that the set in (15) can be covered by $\lessapprox \delta'^{-1}\delta^{2K\varepsilon}$ balls of radius $\lessapprox \delta^{-K\varepsilon/5}\delta'$. Since E_0 is a $(\delta, 1)_2$ set, we see from (1) that

$$\text{LHS of (15)} \lesssim \delta'^{-1} \delta^{2K\varepsilon} \delta' \delta^{-K\varepsilon/5}.$$

But this contradicts (15) if δ is sufficiently small. This concludes the proof of the proposition. □

From (13) and the above proposition we see that (if K is a large enough absolute constant, and δ is sufficiently small depending on ε, K)

$$|\{(x_0, x_1, x_2) \in E_0 \times E_1 \times E_2 : |x_0 - x_1|, |x_0 - x_2| \in D^*\}| \gtrapprox \delta^3. \tag{16}$$

From (12) we have $|D^*| \lessapprox \delta^{1/2}$. From elementary geometry and a change of variables we have

$$|\{x_0 \in E_0 : |x_0 - x_1|, |x_0 - x_2| \in D^*\}| \lessapprox |D^*|^2$$

for all $x_1 \in E_1$, $x_2 \in E_2$. Integrating this over x_1 and x_2 and comparing with the previous we thus see that $|D^*| \approx \delta^{1/2}$. But then (16) contradicts (4) (with D replaced by D^*), if ε is sufficiently small depending on c_1 and δ sufficiently small depending on ε. The full claim of the proposition follows by a modification of this argument, providing that c_4 is sufficiently small depending on c_1.

5. Ring Conjecture 1.14 implies Discretized Furstenburg Conjecture 1.10

Assume that the Ring Conjecture 1.14 is true for some absolute constant $c_4 > 0$. In this section we show how the Discretized Furstenburg Conjecture 1.10 follows.

The main idea is that R is a half-dimensional ring then $R \times R$ contains a one dimensional set of lines each of which contain half dimensional sets. That many of these lines are parallel seems hardly consequential and we will deal with it by an appropriately chosen projective transformation.

Let $0 < \varepsilon \ll 1$ be fixed. We may assume that δ is sufficiently small depending on ε, since (5) is trivial otherwise, and may assume δ is dyadic as before. Let E, Ω, R_ω be as in the Discretized Furstenburg Conjecture 1.10. Assume for contradiction that

$$|\{(x_0, x_1) \in E \times E : x_1, x_0 \in R_\omega \text{ for some } \omega \in \Omega\}| \gtrapprox \delta^2 \tag{17}$$

We will obtain a contradiction from this, and it will be clear from the nature of the argument that (5) in fact holds for some absolute constant $c_3 > 0$ depending on c_4.

It will be convenient to define the non-transitive relation $\sim$ by defining $x \sim y$ if and only if $x, y \in R_\omega$ for some $\omega \in \Omega$. We also write $x_1, \ldots, x_n \sim y_1, \ldots y_m$ if $x_i \sim y_j$ for all $1 \leq i \leq n$ and all $1 \leq j \leq m$.

From (17) we then have

$$|\{(x_0, x_1) \in E \times E : x_0 \sim x_1\}| \gtrapprox \delta^2. \tag{18}$$

Roughly speaking, the idea will be to find $x_1, x_1' \in E$ and a refinement E'' of E such that $x_0 \sim x_1, x_0 \sim x_1'$ for all $x_0 \in E''$, and such that there are many relations between pairs of points in E''. Then after a projective transformation sending x_1, x_1' to the cardinal points at infinity we can transform E'' to a Cartesian product of two $(\delta, \frac{1}{2})_1$ sets of measure $\approx \delta^{1/2}$, at which point the ring structure of these sets can be easily extracted.

We turn to the details. From (18) and the fact that $|E| \approx \delta$, we see that

$$|\{(x_0, x_1) \in E' \times E : x_0 \sim x_1\}| \gtrapprox \delta^2 \tag{19}$$

where

$$E' = \{x_0 : |\{x_1 \in E : x_0 \sim x_1\}| \approx \delta\}$$

provided the constants are chosen appropriately.

Let C_2 be a large constant to be chosen later, and let E_1 be the set

$$E_1 = \{x_1 \in E : \sum_{\omega \in \Omega} \chi_{R_\omega}(x_1) \leq \delta^{-C_2 \varepsilon} \delta^{-1/2}\}.$$

From Kakeya 2.4 and Chebyshev we have

$$|E \backslash E_1| \lessapprox \delta^{2C_2\varepsilon}\delta$$

and thus

$$|\{(x_0, x_1) \in E' \times (E \backslash E_1) : x_0 \sim x_1\}| \lessapprox \delta^{2C_2\varepsilon}\delta^2.$$

If we then choose C_2 is large enough, and δ is small enough depending on C_2 and ε, we thus see from (19) that

(20) $$|\{(x_0, x_1) \in E' \times E_1 : x_0 \sim x_1\}| \gtrapprox \delta^2.$$

In particular, we have $|E_1| \approx \delta$ as before. Henceforth C_2 is fixed so that (20) applies.

From (20) and Cauchy-Schwarz 2.1 we have

(21) $$|\{(x_0, x_1, x_1') \in E' \times E_1 \times E_1 : x_0 \sim x_1, x_1'\}| \gtrapprox \delta^3.$$

Let C_3 be a large constant to be chosen later.

Lemma 5.1. *If C_3 is large enough, and δ is small enough depending on C_3 and ε, we have*

$$|\{(x_0, x_1, x_1') \in E' \times E_1 \times E_1 : x_0 \sim x_1, x_1'; |(x_1 - x_0) \wedge (x_1' - x_0)| \geq \delta^{C_3\varepsilon}\}| \approx \delta^3.$$

Proof. From (20) it suffices to show that

(22)
$$|\{(x_0, x_1, x_1') \in E' \times E_1 \times E_1 : x_0 \sim x_1, x_1'; |(x_1 - x_0) \wedge (x_1' - x_0)| \leq \delta^{C_3\varepsilon}\}| \lessapprox \delta^{C_3\varepsilon/8}\delta^3.$$

(The constant 8 is non-optimal, but this is irrelevant for our purposes). In order to have

$$|(x_1 - x_0) \wedge (x_1' - x_0)| \leq \delta^{C_3\varepsilon}$$

one must either have $|x_1 - x_1'| \lessapprox \delta^{C_3\varepsilon/2}$, or that $|x_1 - x_1'| \gtrapprox \delta^{C_3\varepsilon/2}$ and x_0 is within $\lessapprox \delta^{C_3\varepsilon/2}$ of the line joining $x_1 - x_1'$.

Let us consider the contribution of the former case. Since E_1 is a $(\delta, 1)_2$ set, we see that each pair (x_0, x_1) contributes a set of measure $\lessapprox \delta^{C_3\varepsilon/2}\delta$ to (22). From Fubini's theorem we thus see that the contribution of this case to (22) is acceptable.

Now let us consider the contribution of the latter case. By Fubini's theorem again it suffices to show that

$$|\{x_0 \in E' : x_0 \in S\}| \lessapprox \delta^{C_3\varepsilon/8}\delta$$

for any strip S of width $\lessapprox \delta^{C_3\varepsilon/2}$.

Fix S. From the definition of E' and Fubini's theorem it suffices to show that

$$|\{(x_0, x_2) \in E' \times E : x_0 \in S, x_2 \sim x_0\}| \lessapprox \delta^{C_3\varepsilon/8}\delta^2.$$

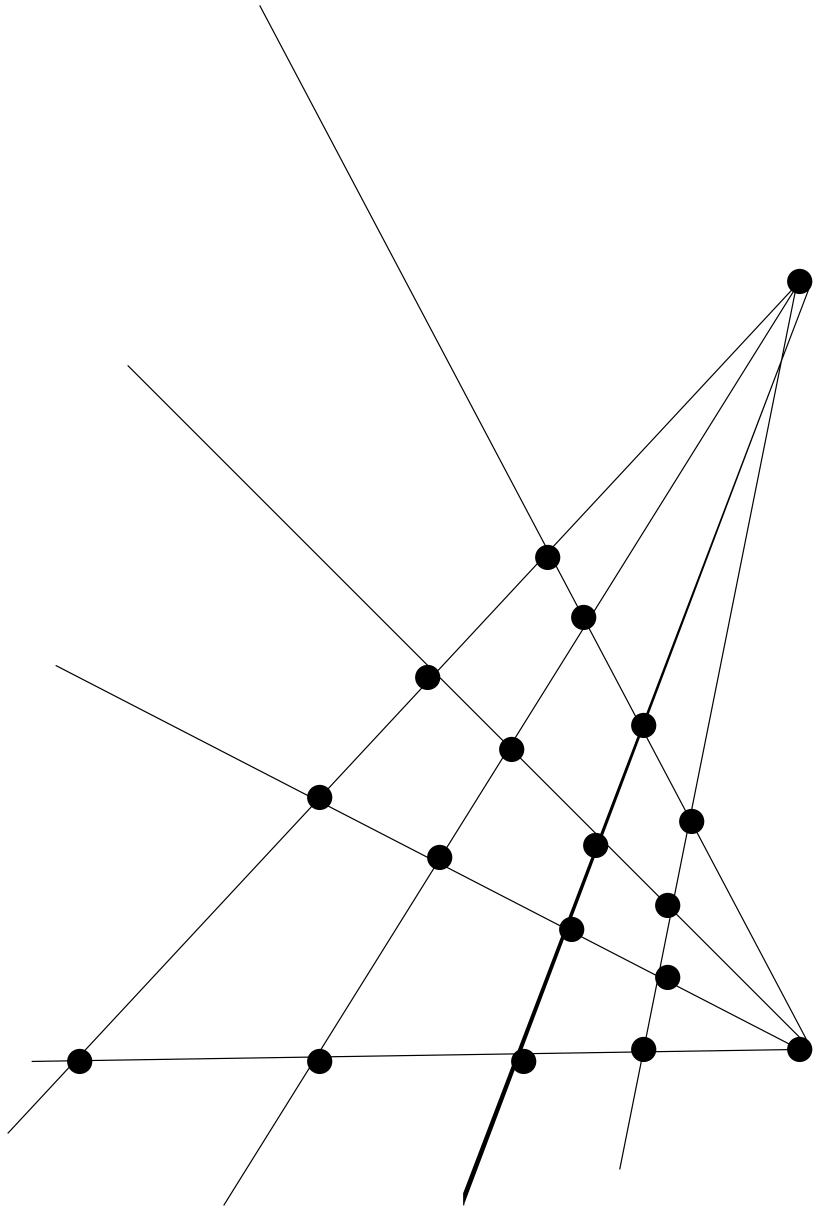

FIGURE 4. A Furstenburg set when viewed from x_1 and x_1'. Note how this resembles a projective transformation of Figure 4.

From the definition of $\sim$, we can estimate the left-hand side by

$$\sum_{\omega \in \Omega} |S \cap R_\omega||R_\omega|.$$

Since R_ω is a $(\delta, \frac{1}{2})_1$ set, we have $|R_\omega| \lessapprox \delta^{3/2}$. Also, if ω makes an angle of $\gtrapprox \delta^{C_3 \varepsilon/4}$ with S we have $|S \cap R_\omega| \lessapprox \delta^{C_3 \varepsilon/8} \delta^{3/2}$ by (1) and elementary geometry, otherwise we may estimate $|S \cap R_\omega| \leq |R_\omega| \lessapprox \delta$. Inserting these estimates into the previous and using the δ-separated nature of the ω, we see that

$$\sum_{\omega \in \Omega} |S \cap R_\omega||R_\omega| \lessapprox \delta^{C_3\varepsilon/8}\delta^2$$

as desired. □

Henceforth C_3 is fixed so that the above lemma applies. We now suppress all explicit mention of C_2, C_3 and absorb these factors into the $\lessapprox$ notation.

From the lemma and the fact that $|E_1| \approx \delta$, we can thus find x_1, x_1' in E_1 such that

$$|\{x_0 \in E' : x_0 \sim x_1, x_1'; |(x_1 - x_0) \wedge (x_1' - x_0)| \approx 1\}| \gtrapprox \delta.$$

Fix x_1, x_1'. Clearly one must have $|x_1 - x_1'| \approx 1$, else the left-hand side is necessarily zero. If we define Q by

$$Q := \{x_0 \in \mathbf{R}^2 : |x_0| \lessapprox 1; |(x_1 - x_0) \wedge (x_1' - x_0)| \approx 1\}$$

and E'' by $E'' := E' \cap Q$, then clearly $|E''| \approx \delta$ if we have chosen the origin appropriately. Also, if we define Ω_1, Ω_1' by

$$\Omega_1 := \{\omega \in \Omega : x_1 \in R_\omega\}, \quad \Omega_1' := \{\omega \in \Omega : x_1' \in R_\omega\},$$

then we have

$$\sum_{\omega \in \Omega_1} \sum_{\omega' \in \Omega_1'} |E'' \cap R_\omega \cap R_{\omega'}| \gtrapprox \delta.$$

From the definition of E_1 we note that

$$\#\Omega_1, \#\Omega_1' \lessapprox \delta^{-1/2}.$$

From the definition of E' we see that

$$\#\{\omega \in \Omega : x_0 \in R_\omega \cap Q\} \gtrapprox \delta^{-1/2}$$

for all $x_0 \in E''$. Integrating this on E'', which has measure $\approx \delta$, we obtain

$$\sum_{\omega \in \Omega} |R_\omega \cap Q \cap E''| \gtrapprox \delta^{1/2}. \tag{23}$$

Let Ω_2 denote those $\omega \in \Omega$ for which the direction of the bounding rectangle for R_ω stays at a distance ≈ 1 from x_1 and x_1'.

Lemma 5.2. *If the constants in the definition of Ω_2 are chosen appropriately, we have*

$$\sum_{\omega \in \Omega_2} |R_\omega \cap Q \cap E''| \gtrapprox \delta^{1/2}. \tag{24}$$

Proof. From elementary geometry we see that

$$\chi^*_{E''}(\omega') \gtrapprox \delta^{-1}|R_\omega \cap Q \cap E''|$$

whenever $|\omega' - \omega| \ll \delta$, where $\chi^*_{E''}$ is the Kakeya maximal function of $\chi_{E''}$. From Kakeya 2.4 we thus have

$$\delta \sum_{\omega \in \Omega} \delta^{-2} |R_\omega \cap Q \cap E''|^2 \lessapprox |E''| \approx \delta.$$

From Cauchy-Schwarz we thus have

$$\sum_{\omega \in \Omega \backslash \Omega_2} |R_\omega \cap Q \cap E''| \lessapprox \#(\Omega \backslash \Omega_2)^{1/2} \delta.$$

If one defines the constants in Ω_2 appropriately, the claim then follows from (23). □

Let $\mathbf{RP}^2$ denote the projective plane, i.e. the points in $\mathbf{R}^3 \backslash \{0\}$ with x identified with tx for all $t \in \mathbf{R} \backslash \{0\}$. We embed $\mathbf{R}^2$ into $\mathbf{RP}^2$ in usual manner, identifying (x, y) with $(x, y, 1)$.

Let $L : \mathbf{RP}^2 \to \mathbf{RP}^2$ be a projective linear transformation which sends x_1 to $(1, 0, 0)$ and x'_1 to $(0, 1, 0)$, but maps Q to a subset of $\mathbf{B}(0, 1)$ with Jacobian ≈ 1 on Q. (This is possible because of the construction of Q). In particular we have

$$|L(E'')| \approx \delta. \tag{25}$$

The set $\bigcup_{\omega \in \Omega_1} R_\omega \cap Q$ stays a distance ≈ 1 from the line joining x_1 and x_2, and is also contained in the union of $\lessapprox \delta^{-1/2}$ rectangles of dimensions about $\delta \times 1$ which pass through x_1. From this fact and some elementary projective geometry we see that

$$L\Big(\bigcup_{\omega \in \Omega_1} R_\omega \cap Q\Big) \subset \mathbf{R} \times B$$

for some δ-discretized set $B \subset [-1, 1]$ with $|B| \approx \delta^{1/2}$. Similarly we have

$$L\Big(\bigcup_{\omega \in \Omega'_1} R_\omega \cap Q\Big) \subset A \times \mathbf{R}$$

for some δ-discretized set $A \subset [-1, 1]$ with $|A| \approx \delta^{1/2}$. Combining these two facts with the definition of E'' we thus have

$$L(E'') \subset A \times B. \tag{26}$$

The sets A and B are already our prototypes for half-dimensional rings. In what follows we refine their geometric properties and establish their algebraic ones.

For all $\omega \in \Omega_2$, let $\tilde{R}_\omega$ denote the set

$$\tilde{R}_\omega := L(R_\omega \cap Q).$$

From the hypothesis on R_ω and some elementary projective geometry we see that $\tilde{R}_\omega$ is a $(\delta, \frac{1}{2})_2$ set which is contained in a rectangle of dimensions $\approx 1 \times \delta$, and whose long side is oriented at an angle of ≈ 1 to the cardinal directions $(0, \pm 1)$, $(\pm 1, 0)$. From (24), (26) we have

$$\sum_{\omega \in \Omega_2} |\tilde{R}_\omega \cap (A \times B)| \gtrapprox \delta^{1/2}. \tag{27}$$

The sets A and B need not be $(\delta, \frac{1}{2})_1$ sets because there is no reason why they should satisfy (1). To rectify this we shall refine A_0, B_0 slightly.

Apply Refinement 2.2, with K a large constant to be chosen shortly, to obtain a covering

$$A \subset A^* \cup \bigcup_{\delta < \delta' \leq 1} A_{\delta'}.$$

From Refinement 2.2, $A_{\delta'}$ can be covered by $\lessapprox \delta^{K\varepsilon} {\delta'}^{-1/2}$ intervals I of length δ'. For each such interval I we have

$$|\tilde{R}_\omega \cap (I \times B)| \lessapprox {\delta'}^{1/2} \delta;$$

this follows from the properties of $\tilde{R}_\omega$, (1), and some elementary geometry. Summing this over I and ω, we obtain

$$\sum_{\omega \in \Omega_2} |\tilde{R}_\omega \cap (A_{\delta'} \times B)| \lessapprox \delta^{K\varepsilon} \delta^{1/2}.$$

Summing this over all δ', we obtain (if K is sufficiently large, and δ sufficiently small depending on K and ε)

$$\sum_{\omega \in \Omega_2} |\tilde{R}_\omega \cap (A^* \times B)| \gtrapprox \delta^{1/2}$$

By breaking A^* up into intervals I of length $\approx \delta$ and arguing as before we see that

$$\sum_{\omega \in \Omega_2} |\tilde{R}_\omega \cap (A^* \times B)| \lessapprox |A^*|;$$

since $|A^*| \leq |A| \lessapprox \delta^{1/2}$, we thus see that $|A^*| \approx \delta^{1/2}$. Also, by Lemma 2.2 we see that A^* is a $(\delta, \frac{1}{2})_1$ set.

By repeating the above argument in the second co-ordinate, we may also find a $(\delta, \frac{1}{2})_1$ set B^* of measure $\approx \delta^{1/2}$ such that

$$\sum_{\omega \in \Omega_2} |\tilde{R}_\omega \cap (A^* \times B^*)| \approx \delta^{1/2}.$$

Let Ω_2' consist of those $\omega \in \Omega_2$ such that

$$|\tilde{R}_\omega \cap (A^* \times B^*)| \gtrapprox \delta^{3/2}. \tag{28}$$

Since $\#\Omega_2 \lessapprox \delta^{-1}$, we thus see that

$$\sum_{\omega \in \Omega_2 \setminus \Omega_2'} |\tilde{R}_\omega \cap (A^* \times B^*)| \leq \frac{1}{2} \sum_{\omega \in \Omega_2} |\tilde{R}_\omega \cap (A^* \times B^*)|$$

if the constants are chosen correctly. We thus have

$$\sum_{\omega \in \Omega_2'} |\tilde{R}_\omega \cap (A^* \times B^*)| \approx \delta^{1/2}.$$

Since $|A^*| \approx \delta^{1/2}$, we can therefore use the pigeonhole principle find an $a \in A^*$ such that

$$\sum_{\omega \in \Omega_2'} |\tilde{R}_\omega \cap (\{a\} \times B^*)| \gtrapprox 1.$$

Fix such an a, and let Ω_2'' consist of those $\omega \in \Omega_2'$ such that

$$|\tilde{R}_\omega \cap (\{a\} \times B^*)| \gtrapprox \delta.$$

By repeating the previous argument, we see that

$$\sum_{\omega \in \Omega_2''} |\tilde{R}_\omega \cap (\{a\} \times B^*)| \gtrapprox 1$$

for suitable choices of constants.

Let C_5 be a constant to be chosen later. Since A^* is a $(\delta, \frac{1}{2})_1$ set, the set $A^* \cap \mathbf{B}(a, \delta^{C_5 \varepsilon})$ can be covered by $\lessapprox \delta^{C_5 \varepsilon/2} \delta^{-1/2}$ intervals I of length δ. By repeating the argument used to refine A and B, we have

$$\sum_{\omega \in \Omega_2''} |\tilde{R}_\omega \cap ((A^* \cap \mathbf{B}(a, \delta^{C_5 \varepsilon}) \times B^*)| \lessapprox \delta^{C_5 \varepsilon/2} \delta^{1/2}.$$

Thus, if C_5 is large enough and δ is small enough depending on C_5 and ε, then

$$\sum_{\omega \in \Omega_2''} |\tilde{R}_\omega \cap ((A^* \backslash \mathbf{B}(a, \delta^{C_5 \varepsilon}) \times B^*)| \approx \delta^{1/2}.$$

Fix C_5, so that all implicit constants may depend on C_5. By the pigeonhole principle again, one can thus find an $a' \in A^*$ such that $|a - a'| \approx 1$ and

$$\sum_{\omega \in \Omega_2''} |\tilde{R}_\omega \cap (\{a'\} \times B^*)| \gtrapprox 1.$$

Fix a'. Let Ω_2''' consist of those $\omega \in \Omega_2''$ such that

$$|\tilde{R}_\omega \cap (\{a'\} \times B^*)| \gtrapprox \delta.$$

Then we have by the same arguments as before that

$$\sum_{\omega \in \Omega_2'''} |\tilde{R}_\omega \cap (\{a'\} \times B^*)| \gtrapprox 1.$$

Since $\tilde{R}_\omega$ is contained in a rectangle of sides $\approx 1 \times \delta$ and making an angle of ≈ 1 with the vertical, we see that

$$|\tilde{R}_\omega \cap (\{a'\} \times B^*)| \lessapprox \delta$$

for all ω. Since $\#\Omega_2''' \leq \#\Omega \lessapprox \delta^{-1}$, we thus see that

$$\#\Omega_2''' \approx \delta^{-1}. \tag{29}$$

Consider the set $X \subset B^* \times B^*$ defined by

$$X := \{(b, b') : (a, b), (a', b') \in \tilde{R}_\omega \text{ for some } \omega \in \Omega_2'''\}.$$

Each ω contributes a set of measure $\gtrapprox \delta^2$ to X. Since the ω are δ-separated and $|a - a'| \approx 1$, we see from elementary geometry that any given point in X can arise from at most $\lessapprox 1$ values of ω. Combining these two facts with (29) we see that

$$|X| \gtrapprox \delta. \tag{30}$$

In particular, X is a refinement of $B^* \times B^*$.

Let C_6 be a large constant to be chosen later. We now wish to find many values of $(b, b') \in X$ and $a'' \in A^*$ such that

$$\frac{a'' - a'}{a - a'} b + \frac{a - a''}{a - a'} b' \in \tilde{B}, \tag{31}$$

where

$$\tilde{B} := B^* + \mathbf{B}(0, C\delta^{1 + C_6 \varepsilon})$$

is a slight enlargement of B^*.

Lemma 5.3. *If C_6 is a sufficiently large constant, and δ is sufficiently small depending on C_6 and ε, then*

$$|\{(b, b', a'') \in X \times A^* : |a'' - a|, |a'' - a'| > \delta^{C_6 \varepsilon}, (31) \textit{ holds}\}| \gtrapprox \delta^{3/2}.$$

Proof. Fix $(b, b') \in X$. From (30) it suffices to show that

$$|\{a'' \in A^* : |a'' - a|, |a'' - a'| > \delta^{C_6 \varepsilon}, (31) \text{ holds}\}| \gtrapprox \delta^{1/2}.$$

From the definition of X and the fact that $\Omega_2''' \subset \Omega_2'$, we can find $\omega \in \Omega_2'$ such that $(a, b), (a', b') \in \tilde{R}_\omega$ and (28) holds. From elementary geometry we see $\tilde{R}_\omega$ stays within $\lessapprox \delta$ of the line

$$\{(a', \frac{a'' - a'}{a - a'} b + \frac{a - a''}{a - a'} b') : a' \in \mathbf{R}\}.$$

Since $\tilde{R}_\omega$ is δ-discretized, we thus have (if C_6 is sufficiently large)

$$|\{a'' \in A^* : (31) \text{ holds}\}| \gtrapprox \delta^{1/2}.$$

The separation conditions $|a'' - a|, |a'' - a'| > \delta^{C_6 \varepsilon}$ are easily imposed by (1), since A^* is a $(\delta, \frac{1}{2})_1$ set. $\square$

Fix C_6; all implicit constants may now depend on C_6. Let T denote the set

$$T = \{\frac{a - a''}{a - a'} : a'' \in A^*\} \cap \{t \in \mathbf{R} : |t| \approx |1 - t| \approx 1\};$$

note that T is a $(\delta, \frac{1}{2})_1$ set. From the above lemma we have

$$|\{(b, b', t) \in B^* \times B^* \times T : (1 - t)b + tb' \in \tilde{B}\}| \gtrapprox \delta^{3/2}.$$

Let B' denote the set of all $b' \in B$ such that

$$|\{(b, t) \in B^* \times T : (1 - t)b + tb' \in \tilde{B}\}| \gtrapprox \delta, \tag{32}$$

From the previous estimate and the fact that $|B^*| \approx \delta^{1/2}$, we see that B' is a refinement of B^* if the constants are chosen correctly. B' is not quite δ-discretized, but this can be easily remedied by introducing the set $B'' := B' + \mathbf{B}(0, \delta)$. B'' is now a $(\delta, \frac{1}{2})$ set, and every element $b' \in B''$ obeys (32) if we enlarge the constant C in the definition of $\tilde{B}$ slightly. In particular, we have

$$|\{(b, b', t) \in B^* \times B'' \times T : (1-t)b + tb' \in \tilde{B}\}| \gtrapprox \delta^{3/2}.$$

Since $|T| \approx \delta^{1/2}$, there thus exists a $t_0 \in T$ such that

$$|\{(b, b') \in B^* \times B'' : (1-t_0)b + t_0 b' \in \tilde{B}\}| \gtrapprox \delta.$$

Applying Perfection 3.4 with A and B replaced by $(1-t_0)B$ and $t_0 B''$, which have measure $\approx \delta^{1/2}$, we can thus find a δ-discretized refinements $(1-t_0)B''''$ and $t_0 B'''$ of $(1-t_0)B$ and $t_0 B''$ respectively such that

$$|(1-t_0)B'''' - t_0 B'''| \approx \delta^{1/2}.$$

From Lemma 3.3 we thus have

$$|t_0 B''' + t_0 B'''| \approx \delta^{1/2}$$

so that

$$|B''' + B'''| \approx \delta^{1/2}. \tag{33}$$

Note that B''' is a δ-discretized refinement of B'' and is therefore a $(\delta, \frac{1}{2})_1$ set with measure $\approx \delta^{1/2}$.

Integrating (32) over all $b' \in B'''$ we have

$$|\{(b, b', t) \in B^* \times B''' \times T : (1-t)b + tb' \in \tilde{B}\}| \gtrapprox \delta^{3/2}.$$

Since $|B^*| \approx \delta^{1/2}$, there thus exists a $b_0 \in B^*$ such that

$$|\{(b', t) \in B''' \times T : (1-t)b_0 + tb' \in \tilde{B}\}| \gtrapprox \delta.$$

Fix b_0. We rewrite the above as

$$|\{(f, t) \in (B''' - b_0) \times T : ft \in \tilde{B} - b_0\}| \gtrapprox \delta.$$

Let C_7 be a constant to be chosen later, and define the set

$$F = \{f \in B''' - b_0 : |f| > \delta^{C_7 \varepsilon}\} + \mathbf{B}(0, \delta).$$

Since B''' is a $(\delta, \frac{1}{2})_1$ set, we have

$$|(B''' - b_0) \backslash F| \lesssim \delta^{C_7 \varepsilon/2} \delta^{1/2}.$$

In particular, we have $|F| \approx \delta^{1/2}$ and

$$|\{(f, t) \in F \times T : ft \in \tilde{B} - b_0\}| \gtrapprox \delta \tag{34}$$

if C_7 is chosen sufficiently large, and δ sufficiently small depending on C_7 and ε.

Fix C_7; all constants may now depend on C_7. From the multiplicative form of Perfection 3.4 and (34) we can thus find a δ-discretized refinement F' of F such that

$$|F'F'| \approx \delta^{1/2}.$$

From the previous we also have $|F' + F'| \lessapprox \delta^{1/2}$. Since F' is a $(\delta, \frac{1}{2})_1$ set of measure $\approx \delta^{1/2}$ contained in some annulus $\mathbf{A}$, we have thus contradicted Conjecture 1.14 if ε is sufficiently small depending on c_4. By modifying the above argument in a routine manner one thus obtains Conjecture 1.10 for c_3 sufficiently small depending on c_4.

6. The discretized Furstenburg Conjecture 1.10 implies the Bilinear Distance Conjecture 1.5

To close the circle of implications and finish the proof of the Main Theorem 1.16 we need to show that the Discretized Furstenburg Conjecture 1.10 implies the Bilinear Ring Conjecture 1.5. This will be done by modifying the argument in Chung, Szemerédi, and Trotter [5], in which the Szemerédi-Trotter theorem was applied to the discrete distance problem. The key geometric fact we use to pass from distances to lines is that if $|x_0 - x_1| = |x_0 - x_2|$, then x_0 lies on the perpendicular bisector of x_1 and x_2. These lines need not point in different directions, but this will be remedied by a generic projective transformation.

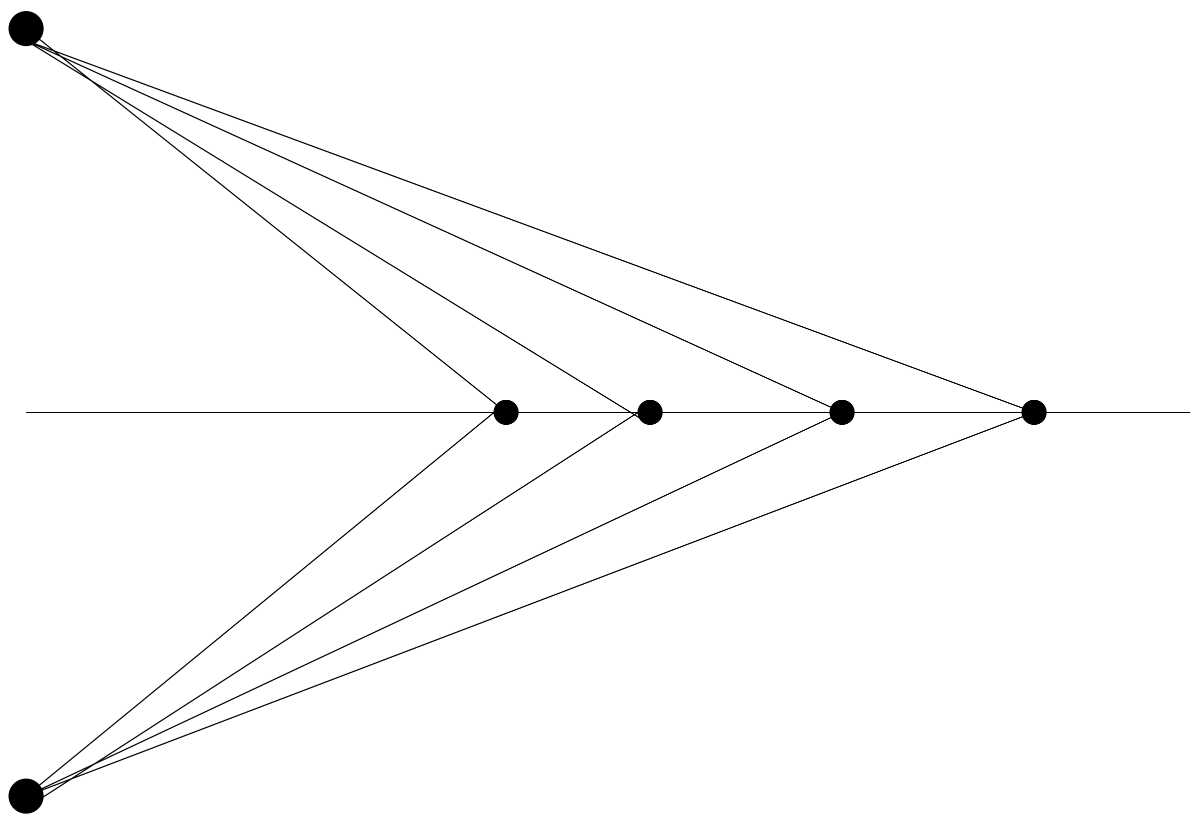

Figure 5. Sets with few distances must concentrate on lines.

Assume that the Discretized Furstenburg Conjecture 1.10 is true for some absolute constant $c_3 > 0$. Let $0 < \varepsilon \ll 1$ be fixed. We may assume that δ is sufficiently small depending on ε, since (4) is trivial otherwise. Let Q_j, E_j, D be as in the Bilinear Distance Conjecture 1.5. Assume for contradiction that

$$|\{(x_0, x_1, x_2) \in E_0 \times E_1 \times E_2 : |x_0 - x_1|, |x_0 - x_2| \in D\}| \gtrapprox \delta^3$$

We will obtain a contradiction from this, and it will be clear from the nature of the argument that (4) in fact holds for some absolute constant $c_1 > 0$ depending on c_3.

Let E_0' denote the set of all $x_0 \in E_0$ such that

$$|\{x_2 \in E_2 : |x_0 - x_2| \in D\}| \geq \delta^{C_8 \varepsilon} \delta,$$

where C_8 is an absolute constant to be chosen later. We have

$$|\{(x_0, x_1, x_2) \in E_0' \times E_1 \times E_2 : |x_0 - x_1|, |x_0 - x_2| \in D\}| \gtrapprox \delta^3 \quad (35)$$

if C_8 is chosen to be sufficiently large and δ sufficiently small depending on C_8 and ε (cf. (19). Since the left hand side is clearly bounded by $|E_0'||E_1||E_2| \approx \delta^2 |E_0'|$, we thus see that $|E_0'| \approx \delta$.

Fix C_8; all implicit constants may now depend on C_8. Since we clearly have

$$|\{x_2 \in E_2 : |x_0 - x_2| \in D\}| \leq |E_2| \lessapprox \delta$$

then we see from (35) that

$$|\{(x_0, x_1) \in E_0' \times E_1 : |x_0 - x_1| \in D\}| \gtrapprox \delta^2.$$

Since D is δ-discretized, we thus have

$$|\{(x_0, x_1, d) \in E_0' \times E_1 \times D : |x_0 - x_1| \in D \cap \mathbf{B}(d, \delta)\}| \gtrapprox \delta^3.$$

Applying Cauchy-Schwarz 2.1 with $A = E_1 \times D$, $B = E_0'$, and $\lambda \approx \delta^{1/2}$ we thus have

$$|\{(x_0, x_0', x_1, d) \in E_0' \times E_0' \times E_1 \times D : |x_0 - x_1|, |x_0' - x_1| \in D \cap \mathbf{B}(d, \delta)\}| \gtrapprox \delta^{9/2}.$$

For fixed x_0, x_0', x_1, the set of all d which contribute to the above set has measure $O(\delta)$, and vanishes unless $|x_0 - x_1| = |x_0' - x_1| + O(\delta)$. Thus we have

$$|\{(x_0, x_0', x_1) \in E_0' \times E_0' \times E_1 : |x_0 - x_1|, |x_0' - x_1| \in D, |x_0 - x_1| = |x_0' - x_1| + O(\delta)\}| \gtrapprox \delta^{7/2}.$$

Let C_9 be an absolute constant to be chosen later.

Lemma 6.1. *We have*

$$|\{(x_0, x_0', x_1) \in E_0' \times E_0' \times E_1 : |x_0 - x_1|, |x_0' - x_1| \in D, |x_0 - x_1| = |x_0' - x_1| + O(\delta); \; |x_0 - x_0'| \leq \delta^{C_9 \varepsilon}\}| \lessapprox \delta^{C_9 \varepsilon / 2} \delta^{7/2}.$$

Proof. Since $|E_0'|, |E_1| \approx \delta$, it suffices to show that

$$|\{x_0 \in E_0' : |x_0 - x_1| = |x_0' - x_1| + O(\delta); |x_0 - x_0'| \leq \delta^{C_9 \varepsilon}\}| \lessapprox \delta^{C_9 \varepsilon / 100} \delta^{3/2}$$

for all x_0', x_1 in E_0', E_1 respectively.

Fix x_0', x_1. By definition of E_0' it suffices to show that

$$|\{(x_0, x_2) \in E_0' \times E_2 : |x_0 - x_1| = |x_0' - x_1| + O(\delta); |x_0 - x_0'| \leq \delta^{C_9 \varepsilon}; |x_0 - x_2| \in D\}| \lessapprox \delta^{C_9\varepsilon/100} \delta^{5/2}.$$

Since $|E_2| \approx \delta$, it thus suffices to show that

(36)

$$|\{x_0 \in E_0' : |x_0 - x_1| = |x_0' - x_1| + O(\delta); |x_0 - x_0'| \leq \delta^{C_9\varepsilon}; |x_0 - x_2| \in D\}| \lessapprox \delta^{C_9\varepsilon/100} \delta^{3/2}$$

for all $x_2 \in E_2$.

Fix x_2. The set in (36) is contained in an annular arc of thickness $O(\delta)$, angular width $O(\delta^{C_9\varepsilon})$, and radius ≈ 1 centered at x_1. From (3) and elementary geometry that the possible values of $|x_0 - x_2|$ thus lie in an interval of length $\lessapprox \delta^{C_9\varepsilon}$. Since D is a $(\delta, \frac{1}{2})_1$ set, we thus see that the possible values of $|x_0 - x_2|$ are contained in the union of $\lessapprox \delta^{C_9\varepsilon/2}\delta^{-1/2}$ intervals of length δ. From (3) and elementary geometry we thus see that the set in (36) can be covered by $\lessapprox \delta^{C_9\varepsilon/2}\delta^{-1/2}$ balls of radius δ. The claim follows. □

If we choose C_9 to be sufficiently large, and δ is sufficiently small depending on C_9 and ε, we thus have from the above that

$$|\{(x_0, x_0', x_1) \in E_0' \times E_0' \times E_1 : |x_0 - x_1|, |x_0' - x_1| \in D; |x_0 - x_1| = |x_0' - x_1| + O(\delta); |x_0 - x_0'| \approx 1\}| \gtrapprox \delta^{7/2},$$

where the implicit constants can now depend on C_9. Since $|E_0'| \approx \delta$, we can thus find $x_0 \in E_0'$ such that

$$|\{(x_0', x_1) \in E_0' \times E_1 : |x_0 - x_1|, |x_0' - x_1| \in D; |x_0 - x_1| = |x_0' - x_1| + O(\delta); |x_0 - x_0'| \approx 1\}| \gtrapprox \delta^{5/2}.$$

Fix this x_0. Let E_0'' denote the set

$$E_0'' := \{x_0' \in E_0' : |x_0 - x_0'| \approx 1\}.$$

For each $x_0' \in E_0''$, let $R[x_0']$ denote the set

$$R[x_0'] := \{x_1 \in E_1 : |x_0 - x_1| \in D; |x_0 - x_1| = |x_0' - x_1| + O(\delta)\}.$$

We thus have

$$\int_{E_0''} |R[x_0']| \, dx_0' \gtrapprox \delta^{5/2}. \tag{37}$$

From elementary geometry, and the fact that Q_0 and Q_1 have separation ≈ 1, we see that $R[x_0']$ is contained in a rectangle $\overline{R}[x_0']$ of dimensions $\approx 1 \times \delta$. Since D is a $(\delta, \frac{1}{2})_1$ set, it is easy to see from elementary geometry that $R[x_0']$ is a $(\delta, \frac{1}{2})_2$ set. In particular, we have $|R[x_0']| \lessapprox \delta^{3/2}$, which implies from (37) that $|E_0''| \gtrapprox \delta$. Since $|E_0| \approx \delta$, we thus have $|E_0''| \approx \delta$. From (37) again we thus see that

$$|\{x_0' \in E_0'' : |R[x_0']| \approx \delta^{3/2}\}| \approx \delta$$

for suitable choices of constants. In particular, we can find a δ-separated set $\Sigma \subset E_0''$ such that $\#\Sigma \approx \delta^{-1}$ and $|R[x_0']| \approx \delta^{3/2}$ for all $x_0' \in \Sigma$.

The sets $R[x_0']$ resemble the sets R_ω in the hypothesis of Conjecture 1.10, but their orientations need not be δ-separated. To remedy this we apply a projective linear transformation.

To find the right transformation to use, we first must isolate a line in $\mathbf{R}^2$ in which the (extensions of) $\overline{R}[x_0']$ are well-separated. To make this precise we apply some normalizations. By a rescaling we may let Q_1 be the square $[0,1] \times [0,1]$, and by a refinement we may assume that the the direction of the $\overline{R}[x_0']$ are within $\pi/4$ of the direction $(1,0)$. In particular, if we extend the long side of $\overline{R}[x_0']$ to have length 10, it will intersect the strip $[2,3] \times \mathbf{R}$ in a parallelogram $P[x_0']$ of thickness $\approx \delta$ and slope $O(1)$.

We now apply

Lemma 6.2. *We have*

$$\| \sum_{x_0' \in \Sigma} \chi_{P[x_0']} \|_2^2 \lessapprox 1.$$

Proof. We shall use Córdoba's argument, using (1) for E_0 as a substitute for the direction-separation property. Expand out the left-hand side as

$$\sum_{x_0' \in \Sigma} \sum_{x_0'' \in \Sigma} |P[x_0'] \cap P[x_0'']|.$$

Since $\#\Sigma \approx \delta^{-1}$, it thus suffices to show that

$$\sum_{x_0' \in \Sigma} |P[x_0'] \cap P[x_0'']| \lessapprox \delta$$

for all $x_0'' \in \Sigma$.

Fix x_0''. The quantity $|P[x_0'] \cap P[x_0'']|$ can vary from 0 to $\approx \delta$. We need only consider the contribution when $\delta^2 \lessapprox |P[x_0'] \cap P[x_0'']| \lessapprox \delta$, since the remaining contribution is trivial to handle. By dyadic pigeonholing and absorbing the logarithmic factor into the $\lessapprox$ symbol, it suffices to show the distributional estimate

$$\#\{x_0' \in \Sigma : |P[x_0'] \cap P[x_0'']| \approx \sigma\delta\} \lessapprox \frac{1}{\sigma}$$

for all dyadic $\delta \lessapprox \sigma \lessapprox 1$.

Fix σ. The set $P[x_0']$ lies within $\lessapprox \delta$ of the perpendicular bisector of x_0' and x_0, and within $\lessapprox 1$ of x_0' and x_0, which are themselves separated by ≈ 1. Similarly for $P[x_0'']$. From elementary geometry we thus see that

$$|P[x_0'] \cap P[x_0'']| \approx \sigma\delta \implies |x_0' - x_0''| \lessapprox \delta/\sigma.$$

Since x_0', x_0'' lie within a δ-separated subset of E_0, which is a $(\delta, 1)_2$ set, we see from (1) that

$$\#\{x_0' \in \Sigma : |x_0' - x_0''| \lessapprox \delta/\sigma\} \lesssim \frac{1}{\sigma}.$$

The claim follows. □

To complement this L^2 bound we have the trivial L^1 bound

$$\| \sum_{x_0' \in \Sigma} \chi_{P[x_0']} \|_1 = \sum_{x_0' \in \Sigma} |P[x_0']| \approx \delta^{-1}\delta = 1.$$

From Hölder's inequality we thus see that $\sum_{x_0' \in \Sigma} \chi_{P[x_0']}$ must be supported on a set of measure $\gtrapprox 1$, so that

$$|\bigcup_{x_0' \in \Sigma} P[x_0']| \gtrapprox 1.$$

Since the set in the left-hand side is contained in the strip $[2,3] \times R$, we can thus find a $2 \leq x \leq 3$ such that

$$|\{y : (x,y) \in \bigcup_{x_0' \in \Sigma} P[x_0']\}| \gtrapprox 1.$$

Fix this x. Each $x_0' \in \Sigma$ contributes an interval of length $\approx \delta$ to the above set. Thus we can find a refinement Σ' of Σ such that the sets $\{y : (x,y) \in P[x_0']\}$ are separated by $\gg \delta$.

Let L be a projective transformation which sends the line $\{x\} \times \mathbf{R}$ to the line at infinity, but maps $[0,1] \times [0,1]$ to a bounded set and has Jacobian ≈ 1 on $[0,1] \times [0,1]$. Thus $L(E_1)$ is a $(\delta, 1)_2$ set with measure $\approx \delta$.

For each $x_0' \in \Sigma'$, we see from elementary projective geometry we see that the sets $L(R[x_0'])$ are $(\delta, \frac{1}{2})_2$ sets contained in a rectangle of dimensions $\approx 1 \times \delta$, and the orientation $\omega = \omega(x_0')$ of these rectangles are δ-separated as x_0' varies along Σ'. Write Ω for the set of all the orientations ω arising in this manner, so that $\#\Omega \approx \delta^{-1}$, and write $R_\omega := L(R[x_0'])$ for all $x_0' \in \Sigma'$. Also write $E := L(E_1)$. Since R_ω is a $(\delta, \frac{1}{2})_2$ set and $|R_\omega| \approx \delta^{3/2}$, we have

$$|\{(x_0, x_1) \in R_\omega : |x_0 - x_1| \approx 1\}| \gtrapprox \delta^3$$

for appropriate choices of constants. For any fixed x_0, x_1 with $|x_0 - x_1| \approx 1$, there are at most $\lessapprox 1$ values of ω for which (x_0, x_1) is contained in the above set, thanks to the δ-separation of the ω. Since $R_\omega \subset E$, we may thus sum the above estimate in ω to obtain

$$|\{(x_0, x_1) \in E : x_0, x_1 \in R_\omega \text{ for some } \omega \in \Omega; |x_0 - x_1| \approx 1\}| \gtrapprox \delta^2.$$

But this contradicts (5) if ε is sufficiently small depending on c_3 and δ is sufficiently small depending on ε. By modifying the above argument in a routine manner one thus obtains the Bilinear Distance Conjecture 1.5 for c_1 sufficiently small depending on c_3. This completes the proof of the Main Theorem 1.16.

7. Discretization of fractals

In order to pass from the δ-discretized Bilinear Distance conjecture 1.5 and Discretized Furstenburg conjecture 1.10 to their respective continuous analogues the Distance Conjecture 1.4 and the Furstenburg conjecture 1.9 we will need some tools to cover an α-dimensional set in $\mathbf{R}^n$ by $(\delta, \alpha + C\varepsilon)_n$ sets for various values of δ.

We begin this section by recalling the definition of Hausdorff dimension.

Definition 7.1. Let $\alpha > 0$. For any bounded set E and $c > 0$, we define the *Hausdorff content* $h_{\alpha,c}(E)$ to be the infimum of the quantity

$$\sum_{i \in I} r_i^\alpha$$

where $\{\mathbf{B}(x_i, r_i)\}_{i \in I}$ ranges over all collections of balls of radii $r_i < c$ which cover E.

$$\dim(E) := \inf\{\alpha : \inf_{c>0} h_{\alpha,c}(E) = 0\} = \sup\{\alpha : \sup_{c>0} h_{\alpha,c}(E) = +\infty\}.$$

Definition 7.2. Let $\{X_\alpha\}_{\alpha \in A}$ be a countable collection of sets. We say that the X_α *strongly cover* E if each point in E is contained in infinitely many sets X_α.

We shall require a variant of the Borel-Cantelli lemma for Hausdorff content.

Lemma 7.3. *Let $0 < \alpha \leq n$, and let $X_i \subset \mathbf{R}^n$ for $i \in \mathbf{Z}$ be such that*

$$\sum_{i=1}^{\infty} h_{\alpha,c}(X_i) < \infty$$

for some $c > 0$. Suppose also that the X_i strongly cover a set E. Then $\dim(E) \leq \alpha$.

Proof. For any integer N, we have $E \subset \bigcup_{i>N} X_i$. Since Hausdorff content is sub-additive, we thus have

$$h_{\alpha,c}(E) \leq \sum_{i>N} h_{\alpha,c}(X_i).$$

The claim then follows by letting $N \to \infty$. □

We can now prove a covering lemma, which is the main result of this section. For technical reasons it will be convenient to not work with dyadic δ as we have done in the past, but move to a much sparser range of scales, namely the hyper-dyadic scales (cf. [3]). More precisely:

Definition 7.4. Let $0 < \varepsilon \ll 1$ be given. We call a number *hyper-dyadic* if it is of the form $2^{-\lfloor (1+\varepsilon)^k \rfloor}$ for some integer $k \geq 0$, where $\lfloor x \rfloor$ is the integer part of x. We call a cube *hyper-dyadic* if it is dyadic and its side-length is hyper-dyadic.

Note that there are at most C_ε hyper-dyadic numbers between δ and δ^{100} for any choice of δ, in contrast to $C \log(1/\delta)$ in the dyadic regime. This improved bound will be important in the proof of Theorem 1.6.

Lemma 7.5. *Let $0 < \varepsilon \ll 1$, $0 < \alpha < n$, and let E be a compact subset of $\mathbf{R}^n$.*

- *If* $\dim(E) \leq \alpha$, *then one can associate a $(\delta, \alpha)_n$ set X_δ to each hyper-dyadic δ such that the X_δ strongly cover E.*
- *Conversely, if C is sufficiently large and there is a $(\delta, \alpha - C\varepsilon)_n$ set X_δ for each hyper-dyadic δ such that the X_δ strongly cover E, then* $\dim(E) \leq \alpha$.

Proof. We first prove the latter claim. Since X_δ is a $(\delta, \alpha - C\varepsilon)_n$ set we can cover it (if the constants are chosen appropriately) by about $\delta^{-\alpha+\varepsilon}$ balls of radius δ, so that

$$h_{\alpha,1}(X_\delta) \leq C\delta^\varepsilon.$$

The claim then follows from Lemma 7.3.

Now we show the former claim. Fix E. For every hyper-dyadic number c, we can find a collection $\{\mathbf{B}(x_{c,i}, r_{c,i})\}_{i \in I_c}$ of balls covering E such that $r_{c,i} < c$ and

$$\sum_{i \in I_c} r_{c,i}^{\alpha + C\varepsilon} \ll 1. \tag{38}$$

By reducing the constant C slightly we may assume that the $r_{c,i}$ are hyper-dyadic.

For each hyper-dyadic r, let $Y_{c,r}$ denote the set

$$Y_{c,r} := \bigcup_{i \in I_c : r_{c,i} = r} \mathbf{B}(x_{c,i}, r_{c,i}).$$

Clearly the sets $Y_{c,r}$ strongly cover E as c, r both vary.

Fix c, r, and let $\mathbf{Q}_{c,r}$ be a collection of hyper-dyadic cubes Q of side-length at least r which cover $Y_{c,r}$ and which minimize the quantity

$$\sum_{Q \in \mathbf{Q}_{c,r}} l(Q)^\alpha,$$

where $l(Q)$ denotes the side-length of Q. Such a minimizer exists since there are only a finite number of hyper-dyadic cubes which are candidates for inclusion in $\mathbf{Q}_{c,r}$. From (38) one can cover $Y_{c,r}$ by at most $r^{-\alpha-\varepsilon}$ cubes of side-length r, hence

$$\sum_{Q \in \mathbf{Q}_{c,r}} l(Q)^\alpha \leq C r^{-\varepsilon}. \tag{39}$$

In particular, we have $l(Q) \leq C r^{-\varepsilon/\alpha}$ for all $Q \in \mathbf{Q}_{c,r}$.

From the construction of $\mathbf{Q}_{c,r}$ we see that the Q are all disjoint, and for all hyper-dyadic cubes I we have

$$\sum_{Q \in \mathbf{Q}_{c,r} : Q \subset I} l(Q)^\alpha \leq l(I)^\alpha \tag{40}$$

since we could otherwise remove those cubes in I from $\mathbf{Q}_{c,r}$ and replace them with I, contradicting minimality.

For each dyadic $r \leq \delta \leq C r^{-\varepsilon/\alpha}$, let $X_{\delta,c,r}$ denote the set

$$X_{\delta,c,r} := \Big(\bigcup_{Q \in \mathbf{Q}_{c,r} : l(Q) = \delta} Q\Big) + \mathbf{B}(0, \delta).$$

Clearly $X_{\delta,c,r}$ is a δ-discretized set. From (40) we see that $X_{\delta,c,r}$ is in fact a $(\delta, \alpha)_n$ set.

Now define $X_\delta := \bigcup_{c,r} X_{\delta,c,r}$. From the constraints $r < c$ and $\delta < Cr^{-\varepsilon/\alpha}$ we see that there are at most $C\log(1/\delta)^2$ pairs (c, r) associated to each δ. Hence X_δ is also a $(\delta, \alpha)_n$ set. By construction we see that the X_δ strongly cover E, and so we are done. □

8. The Discretized Furstenburg conjecture implies the Furstenburg problem.

We now prove Theorem 1.11. Suppose that the Discretized Furstenburg Conjecture 1.10 holds for some $c_3 > 0$, and let K be a $\frac{1}{2}$-set using the notation of the introduction. Let $0 < \varepsilon \ll c_2 \ll c_3^2$ be constants to be chosen later. Assume for contradiction that K has Hausdorff dimension less than $1 + c_2$.

By Lemma 7.5, we may find a $(\delta, 1 + c_2)_2$ set X_δ for each hyper-dyadic δ such that the X_δ strongly cover K.

If $\omega \in S^1$ is a direction, we call ω *bad with respect to* δ if one can find a line l in the direction ω such that

$$h_{1/2-c_2,1}(l \cap X_\delta) \geq \delta^{c_2}, \tag{41}$$

The main estimate we need is

Lemma 8.1. *For all hyper-dyadic δ, we have*

$$|\{\omega \in S^1 : \omega \text{ is bad with respect to } \delta\}| \lessapprox C_{c_2}\delta^{Cc_2} \tag{42}$$

if c_2 is sufficiently small with respect to c_3^2.

Proof. The proof of trivial if δ is large, so we will assume that δ is sufficiently small depending on c_2, c_3.

From Kakeya 2.4 with $f := \chi_{X_\delta}$ and Chebyshev's inequality we have

$$\{\omega \in S^1 : (\chi_{X_\delta})^*_\delta(\omega) > \delta^{-c_2}\delta^{1/2}\} \lessapprox \delta^{Cc_2}.$$

Thus to show (42) it suffices to show that

$$|\{\omega \in S^1 : \omega \text{ is bad with respect to } \delta, (\chi_{X_\delta})^*_\delta(\omega) < \delta^{-c_2}\delta^{1/2}\}| \lessapprox C_{c_2}\delta^{Cc_2}. \tag{43}$$

Suppose for contradiction that (43) failed. Let Ω be a maximal δ-separated subset in the set in (43); we thus have

$$\#\Omega \gtrapprox C_{c_2}\delta^{Cc_2}\delta^{-1}. \tag{44}$$

By construction, for each $\omega \in \Omega$ we can find a line l_ω in the direction ω such that (41) holds. Let R_ω denote the set $(l_\omega + \mathbf{B}(0, \delta)) \cap X_\delta$. From the construction of Ω we thus have

$$|R_\omega| \lessapprox \delta(\chi_{X_\delta})^*_\delta(\omega) \lessapprox \delta^{-Cc_2}\delta^{3/2}. \tag{45}$$

Let $\mathbf{Q}$ be a collection of squares Q of side-length $l(Q) \geq \delta$ which covers R_ω and which minimizes the quantity

$$\sum_{Q \in \mathbf{Q}} l(Q)^{1/2-\sqrt{c_2}}.$$

As in the proof of Lemma 7.5, a minimizer $\mathbf{Q}$ exists and the squares in Q are disjoint and satisfy

$$\sum_{Q \in \mathbf{Q}: Q \subset I} l(Q)^{1/2-\sqrt{c_2}} \leq l(I)^{1/2-\sqrt{c_2}} \tag{46}$$

for all squares I. Also, for all $Q \in \mathbf{Q}$ we have

$$|Q \cap R_\omega| \gtrapprox \delta^2 (l(Q)/\delta)^{1/2-\sqrt{c_2}}$$

since otherwise we could replace Q by all the δ-cubes contained in Q, contradicting the minimality of $\mathbf{Q}$. Summing this over all Q we obtain

$$|R_\omega| \gtrapprox \delta^{3/2+\sqrt{c_2}} \sum_{Q \in \mathbf{Q}} l(Q)^{1/2-\sqrt{c_2}}.$$

From (45) we thus obtain

$$\sum_{Q \in \mathbf{Q}} l(Q)^{1/2-\sqrt{c_2}} \lessapprox \delta^{-\sqrt{c_2}-Cc_2}.$$

We thus have

$$\sum_{Q \in \mathbf{Q}: l(Q) > \delta^{1-A\sqrt{c_2}}} l(Q)^{1/2-c_2} \lessapprox \delta^{(1-A\sqrt{c_2})(\sqrt{c_2}-c_2)} \delta^{-\sqrt{c_2}-Cc_2}$$

If we choose A sufficiently large, we thus have (for δ sufficiently small)

$$\sum_{Q \in \mathbf{Q}: l(Q) > \delta^{1-A\sqrt{c_2}}} l(Q)^{1/2-c_2} \ll \delta^{c_2}.$$

In particular, we have

$$h_{1/2-c_2,1}\Big(\bigcup_{Q \in \mathbf{Q}: l(Q) > \delta^{1-A\sqrt{c_2}}} Q\Big) \ll \delta^{c_2}.$$

On the other hand, since ω is bad with respect to δ, we have

$$h_{1/2-c_2,1}(R_\omega) \geq h_{1/2-c_2,1}(l_\omega \cap X_\delta) \geq \delta^{c_2}.$$

Thus, if we let R'_ω denote the set

$$R'_\omega := \Big(R_\omega \setminus \bigcup_{Q \in \mathbf{Q}: l(Q) > \delta^{1-A\sqrt{c_2}}} Q\Big) + \mathbf{B}(0,\delta)$$

then we have

$$h_{1/2-c_2,1}(R'_\omega) \gtrapprox \delta^{c_2}.$$

Since R'_ω is δ-discretized, we have in particular that

$$|R'_\omega| \gtrapprox \delta^{3/2+Cc_2}.$$

The set R'_ω is covered by the dilates of those cubes $Q \in \mathbf{Q}$ for which $l(Q) \leq \delta^{1-A\sqrt{c_2}}$. From this and (46) we see that R'_ω is a $(\delta, 1/2)_2$ set but with ε replaced by $A\sqrt{c_2}$. From (5) we thus have

$$|\{(x_0, x_1) \in K \times K : x_1, x_0 \in R'_\omega \text{ for some } \omega \in \Omega\}| \lessapprox \delta^{2+c_3-C\sqrt{c_2}}.$$

On the other hand, from Separation 2.3 we have

$$|\{(x_0, x_1) \in R'_\omega : |x_0 - x_1| \gtrapprox \delta^{CA\sqrt{c_2}}\}| \gtrapprox \delta^{3+CA\sqrt{c_2}}.$$

Summing this on ω using (44) and noting that each (x_0, x_1) can be in at most $\lessapprox \delta^{-CA\sqrt{c_2}}$ of the above sets, we obtain

$$|\{(x_0, x_1) \in K \times K : x_1, x_0 \in R'_\omega \text{ for some } \omega \in \Omega\}| \gtrapprox C_{c_2}\delta^{2+CA\sqrt{c_2}}.$$

If c_2 is sufficiently small with respect to c_3^2 we obtain the desired contradiction, if δ is sufficiently small. □

If ε is chosen sufficiently small depending on c_2, then the left-hand side of (42) is thus summable in δ. By the Borel-Cantelli lemma (for Lebesgue measure) we can thus find a direction ω which is only bad with respect to a finite number of hyper-dyadic δ. In particular we have

$$\sum_\delta h_{1/2-c_2,1}(l \cap X_\delta) < \infty$$

for all lines l parallel to ω. Since the $l \cap X_\delta$ strongly covers $l \cap K$, we thus see from Lemma 7.3 that $\dim(l \cap K) < 1/2$ for all l parallel to ω. But this contradicts the assumption that K is a $\frac{1}{2}$-set. ■

We remark that a similar result obtains for all β-sets providing that β is sufficiently close to $\frac{1}{2}$ (depending on c_3).

9. The Bilinear Distance conjecture implies the Falconer Distance Conjecture

We now prove Theorem 1.6. Suppose that the Bilinear Distance Conjecture 1.5 holds for some $c_1 > 0$. Let $0 < \varepsilon \ll c_0 \ll c_1$ be constants to be chosen later. Assume for contradiction that one can find a compact set K with dimension $\dim(K) \geq 1$ such that $\dim(\text{dist}(K)) \leq 1/2 + c_0$.

By Frostman's lemma [10] we may find a probability measure μ supported on K such that

$$\mu(\mathbf{B}(x, r)) \leq C_\varepsilon r^{1-\varepsilon} \tag{47}$$

for all balls $\mathbf{B}(x, r)$. Fix this μ.

By Lemma 7.5 we may find a $(\delta, 1/2 + c_0)_1$ set D_δ for each hyper-dyadic δ such that the D_δ strongly cover $\text{dist}(K)$. If one then defines

$$X_\delta := \{(x, y) \in K \times K : |x - y| \in D_\delta\}$$

then the X_δ strongly cover $K \times K$.

If it were not for the bilinear formulation of Conjecture 1.5, one could hope to prove a bound like

$$\mu(X_\delta) \lessapprox \delta^{C^{-1}c_0}, \tag{48}$$

if $c_0 \ll c_1$, which would allow us to obtain a contradiction from the Borel-Cantelli lemma. These types of bounds however are not achievable because of the counterexample (2). Furthermore, it is possible for K to contain obstructions like (2) at infinitely many scales. Fortunately, one can show that it is not possible for too many pairs $x, y \in K$ to be simultaneously contained in sets like (2) at infinitely many scales, which allows one to proceed. Of course, one has to make the notion of "looking like (2)" precise, which causes some unpleasant technicalities.

We begin by converting (4) to a bilinear variant of (48).

Lemma 9.1. *If c_0 is sufficiently small depending on c_1, and δ is sufficiently small, then*

(49)

$$\mu^3(\{(x_0,x_1,x_2) : (x_0,x_1),(x_0,x_2) \in X_\delta; |(x_1-x_0)\wedge(x_2-x_0)| \geq \delta^{C^{-1}c_0}\}) \leq C_{\varepsilon,c_0}\delta^{C^{-1}c_0}.$$

where μ^3 is product measure on $K \times K \times K$.

Proof. Let c_5 be a constant to be chosen later. We shall show that

$$\begin{aligned}\mu^3(\{(x_0,x_1,x_2) :&(x_0,x_1),(x_0,x_2) \in X_\delta; |(x_1-x_0)\wedge(x_2-x_0)| \geq \delta^{c_5}\}) \\ &\leq C_{\varepsilon,c_0}(\delta^{C^{-1}c_5} + \delta^{-Cc_5}\delta^{C^{-1}c_0}),\end{aligned} \tag{50}$$

from which (49) follows from a suitable choice of c_5.

Partition $K = K_1 \cup K_2$, where

$$K_1 := \{x \in K : \mu(B(x,\delta)) \geq \delta^{c_5}\delta\},$$

$$K_2 := \{x \in K : \mu(B(x,\delta)) < \delta^{c_5}\delta\}.$$

Let us first deal with the contribution to (50) of the case when at least one of x_0, x_1, x_2 is in K_2. By Fubini's theorem and symmetry it suffices to show that

$$\mu^2(\{(x_0,x_1) : x_0 \in E_2, (x_0,x_1) \in X_\delta\}) \leq C_{\varepsilon,c_0}\delta^{c_5/10}.$$

By Cauchy-Schwarz 2.1 it suffices to show that

$$\mu^3(\{(x_0,x_1,x_2) : x_0 \in E_2, (x_0,x_1),(x_0,x_2) \in X_\delta\}) \lessapprox \delta^{c_5/5}.$$

Let us first consider the contribution of the case $|x_1 - x_2| \lessapprox \delta^{c_5/5}$. For each x_0, x_1, the set of x_2 which contribute to the above expression has measure $O(\delta^{c_5/5})$ by (47), and so this contribution is acceptable by Fubini's theorem. It thus remains to show

$$\mu^3(\{(x_0,x_1,x_2) : x_0 \in E_2, (x_0,x_1),(x_0,x_2) \in X_\delta, |x_1-x_2| \gtrapprox \delta^{c_5/5}\}) \lessapprox \delta^{c_5/5}.$$

By Fubini's theorem again, it suffices to show that

$$\mu(\{x_0 \in E_2 : (x_0, x_1), (x_0, x_2) \in X_\delta\}) \lessapprox \delta^{c_5/5}$$

for all $x_1, x_2 \in E$ such that $|x_1 - x_2| \gtrapprox \delta^{c_5/5}$.

Fix x_1, x_2. By (47) again, it suffices to show that

$$\mu(\{x_0 \in E_2 : |x_0 - x_1|, |x_0 - x_2| \in D_\delta; |x_0 - x_1|, |x_0 - x_2| \gtrapprox \delta^{c_5/5}\}) \lessapprox \delta^{c_5/5}.$$

The set D_δ can be covered by $\lessapprox \delta^{-1/2}$ intervals of length δ. Let $\mathbf{I}$ denote the collection of those intervals I in this cover such that $\operatorname{dist}(0, I) \gtrapprox \delta^{c_5/5}$. It suffices to show that

$$\sum_{I,J \in \mathbf{I}} \mu(\{x_0 \in E_2 : |x_0 - x_1| \in I; |x_0 - x_2| \in J\}) \lessapprox \delta^{c_5/5}. \tag{51}$$

For fixed I, J, the set described above is contained in an annular arc of thickness δ, radius $\delta^{c_5/5} \lessapprox r \lessapprox 1$, and angular width bounded by

$$\lessapprox \frac{\delta^{-c_5/10}\delta}{(\delta + |\operatorname{dist}(I, J) - |x_1 - x_2||)^{1/2}}$$

as one can easily compute using elementary geometry. By the construction of E_2 and a simple covering argument, we can thus bound the left-hand side of (51) by

$$\lessapprox \delta^{c_5} \frac{\delta^{-c_5/10}\delta}{(\delta + |\operatorname{dist}(I, J) - |x_1 - x_2||)^{1/2}}.$$

To complete the proof of (51) it thus suffices to show that

$$\sum_{J \in \mathbf{I}} \frac{\delta^{-c_5/10}\delta}{(\delta + |\operatorname{dist}(I, J) - |x_1 - x_2||)^{1/2}} \lessapprox \delta^{1/2}$$

for all $I \in \mathbf{I}$. But this follows easily by dyadically decomposing the J based on $\operatorname{dist}(I, J)$ and noting from (1) that for each k, there are $\lessapprox 2^{k/2}$ intervals J for which $\operatorname{dist}(I, J) \approx 2^k\delta$. Note that any logarithmic factors can be absorbed into the $\lessapprox$ notation.

To conclude the proof of (50) it remains to show that

$$\mu^3(\{(x_0, x_1, x_2) \in K_1 \times K_1 \times K_1 : (x_0, x_1), (x_0, x_2) \in X_\delta;$$
$$|(x_1 - x_0) \wedge (x_2 - x_0)| \geq \delta^{c_5}\}) \leq C_{\varepsilon, c_0}\delta^{-Cc_5}\delta^{C^{-1}c_0}.$$

From the definition of K_1 and (47) we see that K_1 is contained in a $(\delta, \frac{1}{2} + c_5)_2$ set E. From (47) and a covering argument we have

$$\mu^3(\{(x_0, x_1, x_2) \in K_1 \times K_1 \times K_1 : (x_0, x_1), (x_0, x_2) \in X_\delta; |(x_1 - x_0) \wedge (x_2 - x_0)| \geq \delta^{c_5}\})$$
$$\lessapprox \delta^{-3}|\{(x_0, x_1, x_2) \in E \times E \times E : |x_0 - x_1|, |x_0 - x_2| \in D_\delta + B(0, \delta^{1-C\varepsilon});$$
$$|(x_1 - x_0) \wedge (x_2 - x_0)| \geq \frac{1}{2}\delta^{c_5}\}|.$$

The claim then follows from (4). □

Henceforth we assume that c_0 is so small that Lemma 9.1 holds.

We shall use (49) to create a dichotomy, that either (48) holds or that the pairs in X_δ are concentrated in a thin set resembling (2). More precisely, we have

Lemma 9.2. *If C_{11} is a sufficiently large constant, then for each δ we can find an integer N_δ and sets $S_{\delta,1}, \dots, S_{\delta,N_\delta} \subset B(0,C)$ such that*

- *For each δ, each $x \in K$ is contained in at most C sets $S_{\delta,i}$, where C is an absolute constant independent of δ.*
- *Each set $S_{\delta,i}$ is contained in a strip R_i (i.e. a rectangle of infinite length) with width $\delta^{C_{11}^{-1}c_0}$.*
- *For each i one can find a finite set $F_i \subset \mathbf{B}(0,C)$ of points with cardinality*

$$\#F_i \lessapprox \delta^{-Cc_0} \tag{52}$$

  and for each $x \in F_i$ one can associate a collection $A_{i,x,1}, \dots, A_{i,x,M_{i,x}}$ of annuli of thickness $C\delta$, radii which are $\lessapprox 1$ and δ-separated, and center x such that

$$M_{i,x} \lessapprox \delta^{-Cc_0}\delta^{-1/2} \tag{53}$$

  for all $x \in F_i$ and

$$S_{\delta,i} \subset \bigcup_{x\in F_i} \bigcup_{j=1}^{M_{i,x}} A_{i,x,j}. \tag{54}$$

- *We have the estimate*

$$\mu^2(Y_\delta) \lessapprox \delta^{CC_{11}^{-1}c_0}, \tag{55}$$

  where

$$Y_\delta := X_\delta \backslash \bigcup_{i=1}^{N_\delta} S^2_{\delta,i}.$$

Proof. Fix δ. The space of all strips of width $\delta^{C_{11}^{-1}c_0}$ which intersect $\mathbf{B}(0,C)$ is a two-dimensional manifold, which we can endow with a smooth metric $d(\cdot,\cdot)$. Let $R_1, \dots, R_{N_\delta}$ be a maximal $C^{-1}\delta^{C_{11}^{-1}c_0}$-separated subset of this space of strips; note that $N_\delta \lessapprox \delta^{-2C_{11}^{-1}c_0}$.

For each $x \in K$, let $i(x)$ denote the index $1 \leq i \leq N_\delta$ which maximizes the quantity

$$\mu(\{y \in K : x, y \in R_i\}),$$

and for each $1 \leq i \leq N_\delta$, define $T_{\delta,i}$ to be the set

$$T_{\delta,i} := \{x \in K : d(R_i, R_{i(x)}) \leq C\delta^{C_{11}^{-1}c_0}\}.$$

Clearly the sets $T_{\delta,i}$ are contained in R_i and form a finitely overlapping cover of K. We shall show the preliminary estimate

$$\mu^2(X_\delta \backslash \bigcup_{i=1}^{N_\delta} T^2_{\delta,i}) \lessapprox \delta^{CC_{11}^{-1}c_0}. \tag{56}$$

It suffices to show
$$\mu^2(\{(x_0,x_1) \in X_\delta : d(R_{i(x_0)}, R_{i(x_1)}) \geq C\delta\}) \lessapprox \delta^{CC_{11}{}^{-1}c_0}.$$

From the bounds on N_δ it suffices to show that
$$\mu^2(\{(x_0,x_1) \in X_\delta : i(x_0) = i, i(x_1) = j\}) \lessapprox \delta^{CC_{11}{}^{-1}c_0}$$
for all $1 \leq i,j \leq N_\delta$ such that $d(R_i,R_j) \geq C\delta$.

Fix i, j, and rewrite the above as
$$\int_{i(x_0)=i} \mu(\{x_1 : (x_0,x_1) \in X_\delta, i(x_1) = j\})\, d\mu(x_0) \lessapprox \delta^{CC_{11}{}^{-1}c_0}.$$
By Cauchy-Schwarz it suffices to show that
$$\int_{i(x_0)=i} \mu(\{x_1 : (x_0,x_1) \in X_\delta, i(x_1) = j\})^2\, d\mu(x_0) \lessapprox \delta^{CC_{11}{}^{-1}c_0}.$$
By definition of $i(x_0)$, we have
$$\mu(\{x_1 : (x_0,x_1) \in X_\delta, i(x_1) = j\}) \leq \mu(\{x_2 : (x_0,x_2) \in X_\delta, i(x_2) = i\}),$$
so it suffices to show that
$$\int_{i(x_0)=i} \mu^2(\{(x_1,x_2) : (x_0,x_1),(x_0,x_2) \in X_\delta, i(x_1) = j, i(x_2) = i\})\, d\mu(x_0). \lessapprox \delta^{CC_{11}{}^{-1}c_0}.$$
This will obtain if we can show
$$\mu^3(\{(x_0,x_1,x_2) : x_0,x_2 \in R_i; x_1 \in R_j; (x_0,x_1),(x_0,x_1) \in X_\delta\}) \lessapprox \delta^{CC_{11}{}^{-1}c_0}.$$
We first consider the contribution of the case when $|x_1 - x_0| \leq \delta^{C_{11}{}^{-1}c_0}$. In this case we estimate x_1 integral by (47) and then integrate in the x_0 and x_2 variables to show that the contribution of this case is acceptable. Similarly we can handle the case $|x_2 - x_0| \leq \delta^{C_{11}{}^{-1}c_0}$. Thus it remains to show that
$$\mu^3(\{(x_0,x_1,x_2) : x_0,x_2 \in R_i; x_1 \in R_j; (x_0,x_1),(x_0,x_1) \in X_\delta; |x_1 - x_0|, |x_2 - x_0| > \delta^{C_{11}{}^{-1}c_0}\}) \lessapprox \delta^{CC_{11}{}^{-1}c_0}.$$
Suppose (x_0,x_1,x_2) is in the above set. Since $d(R_i,R_j) > C\delta$, we see from elementary geometry that
$$|(x_1 - x_0) \wedge (x_2 - x_0)| \geq \delta^{CC_{11}{}^{-1}c_0}.$$
Thus the desired claim follows from (49), if C_{11} is sufficiently large.

The $T_{\delta,i}$ have most of the properties that we desire for $S_{\delta,i}$, but need not be covered by a small number of annuli. To remedy this we shall refine $T_{\delta,i}$ slightly.

Fix j and perform the following algorithm. Initialize $S_{\delta,i}$ to be the empty set. If one has
$$\mu(\{y \in T_{\delta,i} \backslash S_{\delta,i} : (x,y) \in X_\delta\}) \lessapprox \delta^{Cc_0} \tag{57}$$
for all $x \in T_{\delta,i}$, we terminate the algorithm. Otherwise, we choose an $x \in T_{\delta,i}$ for which (57) fails, and add the set in (57) to $S_{\delta,i}$. We then repeat this algorithm, continuing to enlarge $S_{\delta,i}$ until (57) is finally satisfied for all $x \in T_{\delta,i}$.

Since each iteration of this algorithm adds a set of measure $\gtrapprox \delta^{Cc_0}$ to $S_{\delta,i}$, this algorithm must terminate after at most $\lessapprox \delta^{-Cc_0}$ steps. Since X_δ is a $(\delta, \frac{1}{2}+c_0)_1$ set, we see that each set of the form (57) can be covered by annuli $A_{i,x,1}, \ldots, A_{i,x,M_x}$ with width δ, radii $\lessapprox 1$ and $C^{-1}\delta$-separated, and center at x. Thus we have the desired covering (7.5).

It remains to prove (55). From (56) and the bounds on N_δ it suffices to show that

$$\mu^2(X_\delta \cap (T^2_{\delta,i} \backslash S^2_{\delta,i})) \lessapprox \delta^{CC_{11}{}^{-1}c_0}$$

for all $1 \leq i \leq N_\delta$. Since

$$T^2_{\delta,i} \backslash S^2_{\delta,i} = T_{\delta,i} \times (T_{\delta,i} \backslash S_{\delta,i}) \cup (T_{\delta,i} \backslash S_{\delta,i}) \times T_{\delta,i}$$

it suffices by symmetry to show that

$$\mu^2(X_\delta \cap (T_{\delta,i} \times (T_{\delta,i} \backslash S_{\delta,i}))) \lessapprox \delta^{CC_{11}{}^{-1}c_0}$$

for all i. But this follows by integrating (57) over all $x \in T_{\delta,i}$. □

Henceforth C_{11} will be assumed large enough so that the above lemma holds.

From (55) we see that $\sum_\delta \mu^2(Y_\delta) < \infty$, if ε is chosen sufficiently small depending on c_0. From the Borel-Cantelli lemma we thus see that for almost every x, y, the pair (x, y) is contained in only finitely many Y_δ. (Here and in the sequel, "almost every" is with respect to μ). Since the X_δ strongly cover $E \times E$, we thus see that (x, y) is contained in infinitely many sets of the form $S^2_{\delta,i}$ for hyper-dyadic δ and $1 \leq i \leq N_\delta$ for almost every x, y.

Suppose $x \in E$, and δ is a hyper-dyadic number. Define $d(x, \delta)$ to be the smallest hyper-dyadic number δ_1 such that

$$\delta_1 > \delta^{C_{11}/c_0} \tag{58}$$

and such that $x \in S_{\delta_1,i_1}$ for some $1 \leq i_1 \leq N_{\delta_1}$, or $d(x, \delta) = +\infty$ if no such δ_1 exists. From the previous observation we thus see that for almost every x, y, there are infinitely many hyper-dyadic δ, δ_1, δ_2 and i such that $x, y \in S_{\delta,i}$, $\delta_1 = d(x, \delta)$, and $\delta_2 = d(y, \delta)$. In particular, we have

$$\sum_{\delta_1}\sum_{\delta_2} \mu^2\{(x, y) \in E \times E : x, y \in S_{\delta,i}, \delta_1 = d(x, \delta), \delta_2 = d(y, \delta) \text{ for some } \delta, i\} = \infty.$$

The desired contradiction then follows immediately (if c_0 is sufficiently small) from

Lemma 9.3. *For all hyper-dyadic δ_1, δ_2 we have (if C_{11} is chosen appropriately)*

$$\mu^2\{(x, y) \in K \times K : x, y \in S_{\delta,i}, \delta_1 = d(x, \delta), \delta_2 = d(y, \delta) \textit{ for some } \delta, i\} \\ \leq C_{c_0,\varepsilon} \min(\delta_1, \delta_2)^{1/4 - Cc_0}.$$

The 1/4 exponent is not optimal, but that is irrelevant for our purposes, since we only need the right-hand side to be summable in δ_1, δ_2.

Proof. Fix δ_1, δ_2; by symmetry we may assume that $\delta_1 \leq \delta_2$. By Fubini's theorem it suffices to show that

$$\mu\{x \in K : x, y \in S_{\delta,i}, \delta_1 = d(x,\delta), \delta_2 = d(y,\delta) \text{ for some } \delta, i\} \leq C_{c_0}\delta_1^{1/4-Cc_0} \tag{59}$$

for all $y \in K$.

Fix y. Since y and δ_2 are fixed, there are significant constraints on the number of δ which can contribute to (59). Indeed, if there are two values of δ, say δ' and δ'', which contribute to (59), then δ' cannot exceed ${\delta''}^{C_{11}/c_0}$ and δ'' cannot exceed ${\delta'}^{C_{11}/c_0}$, due to the presence of (58) in the definition of $d(y,\delta')$, $d(y,\delta'')$. Because δ is constrained to be hyper-dyadic, we thus see that there are at most $C_{C_{11},c_0,\varepsilon}$ values of δ which contribute to (59). Thus it suffices to show (59) for a single value of δ. Since the $S_{\delta,i}$ are finitely overlapping as i varies, we see that for each δ there are at most C values of i which contribute to (59). Hence it suffices to show that

$$\mu\{x \in K : x \in S_{\delta,i}, \delta_1 = d(x,\delta)\} \leq C_{c_0}\delta_1^{1/4-Cc_0} \tag{60}$$

for all δ, i.

Fix δ, i. We may of course assume that (58) holds, else (60) is vacuously true. By definition of $d(x,\delta)$ and the fact that $N_{\delta_1} \lessapprox \delta_1^{-Cc_0}$ it suffices to show that

$$\mu\{x \in K : x \in S_{\delta,i}, x \in S_{\delta_1,i_1}\} \leq C_{c_0}\delta_1^{1/4-Cc_0}$$

for all i_1.

Fix i_1. By (54) and (52) it suffices to show that

$$\sum_{j=1}^{M_{i_1,x_0}} \mu(S_{\delta,i} \cap A_{i_1,x_0,j}) \leq C_{c_0}\delta_1^{1/4-Cc_0}$$

for all $x_0 \in F$.

Fix x_0. The set $S_{\delta,i}$ is contained in a rectangle of width $\delta^{C_{11}{}^{-1}c_0}$, hence contained in a rectangle R of width δ_1 by (58). Let r denote the distance from R to x_0. From elementary geometry we see that $R \cap A_{i_1,x_0,j}$ is the union of two sets, each of which having diameter at most

$$\frac{C\delta}{(\delta + |r_j - r|)^{1/2}}$$

where r_j is the outer radius of $A_{i_1,x_0,j}$. From (47) we thus have

$$\mu(S_{\delta,i} \cap A_{x_0,j}) \leq C\delta^{1-\varepsilon}(\delta + |r_j - r|)^{1/2-\varepsilon}$$

If we arrange the r_j in order of distance from r, we have $|r_j - r| \geq Cj\delta$ since the r_j are δ-separated. Since $\varepsilon \ll c_0$, the claim then follows from (53). $\square$

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