ground water

Issue Paper/

Do a Bit More with Convolution

by Theo N. Olsthoorn[1,2]

Abstract

Convolution is a form of superposition that efficiently deals with input varying arbitrarily in time or space. It works whenever superposition is applicable, that is, for linear systems. Even though convolution is well-known since the 19th century, this valuable method is still missing in most textbooks on ground water hydrology. This limits widespread application in this field. Perhaps most papers are too complex mathematically as they tend to focus on the derivation of analytical expressions rather than solving practical problems. However, convolution is straightforward with standard mathematical software or even a spreadsheet, as is demonstrated in the paper. The necessary system responses are not limited to analytic solutions; they may also be obtained by running an already existing ground water model for a single stress period until equilibrium is reached. With these responses, high-resolution time series of head or discharge may then be computed by convolution for arbitrary points and arbitrarily varying input, without further use of the model. There are probably thousands of applications in the field of ground water hydrology that may benefit from convolution. Therefore, its inclusion in ground water textbooks and courses is strongly needed.

Introduction

Convolution is a well-known and effective superposition method to deal with arbitrary inputs in time and space. In physics, convolution appears wherever linear systems are at hand so that superposition is valid. It is applied in numerous branches of science such as statistics, optics, acoustics, signal processing (e.g., Oppenheim et al. 1997), geophysics, and surface water hydrology (unit hydrograph methods). See, for instance, Oppenheim et al. (1997) for an introduction to convolution.

There is a substantial number of papers showing applications of convolution in ground water hydrology. Nevertheless, convolution is missing in the majority of ground water hydrology textbooks, for example, in Freeze and Cherry (1979), Strack (1989), Dingman (2002), Fitts (2002), Domenico and Schwartz (1998), Todd and Mays (2005), Schwartz and Zhang (2003), and Fetter (2001) and even in books on pumping tests such as Kruseman and De Ridder (1970, 1994).

Other well-known ground water hydrology textbooks like Bear (1972, 1979) and Charbeneau (2000) explain Duhamel's (1797 to 1872) integral (or convolution integral) but do not present it as a practical and general tool for day-to-day application in ground water hydrology.

de Marsily (1986), who did much work on convolution before writing his book, spends only eight lines on it on page 197. A more specialized book like Lee (1999) explicitly deals with convolution, but it may not be so often read by practitioners. Books on analytical solutions to ground water problems like Carslaw and Jaeger (1959) and Bruggeman (1999) present hundreds of analytical solutions for ground water flow problems in terms of convolution without calling them so. In conclusion, convolution is still widely ignored in textbooks in hydrogeology, which causes unawareness of its practical applicability among many ground water hydrologists.

This paper seeks to encourage including convolution in courses and textbooks on hydrogeology as a practical,

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efficient, and widely applicable method of superposition to tackle arbitrary input. The conclusion of Hall and Moench (1972) that "there is a need for a broad consideration of possible applications and a comprehensive discussion on the convolution relation" seems still valid today in the field of ground water hydrology.

Clearly, there have been a substantial number of papers in the field of ground water hydrology that apply convolution. Perhaps the mathematical character of most of them, focusing on Laplace transforms of analytical solutions, may have withheld hydrologists from applying it. As will be shown, this is unjustified; convolution can readily be done with virtually any of the mathematical programs around, including a spreadsheet.

Convolution may be based on analytical solutions, but equally well on a numerical model, on time series analysis, or on any other means by which a so-called impulse response or unit step response can be obtained.

Most papers addressing convolution in our field deal with ground water–surface water interaction. Cooper and Rorabaugh (1963) focused on analytical expressions of the reaction of ground water discharge, bank storage, and heads to a flood. Venetis (1968, 1970) provided a simple derivation for the reaction of the ground water to surface water stage variations. Moench and Kisiel (1970) applied it to stream depletion, and Hall and Moench (1972) added entry resistance between stream and aquifer. Zhang (1992) added the effect of storage in an overlying aquitard. Later, a full set of analytical expressions of ground water–surface water interaction for all previous aquifer types was presented by Barlow and Moench (1998) and Moench and Barlow (2000). These solutions found their way into the USGS programs STLK1 and STWT1 (Barlow and Moench 1998; DeSimone and Barlow 1999) and were illustrated in the paper by Barlow et al. (2000).

Convolution has also been used to obtain aquifer parameters from ground water head responses from fluctuating river stages (e.g., Pinder et al. 1969; Rowe 1960). Parameter estimation by convolution using pumping of large-diameter wells was illustrated by Singh and Gupta (1986), who also provided a program. Knight et al. (2004) and Rassam et al. (2004) applied convolution to the salt discharge into Australian rivers caused by fieldwise irrigation on salt aquifers. Convolution has also been successfully applied in ground water management models to optimize conjunctive use of surface and ground water (Illangsekare and Morel-Seytoux 1986). Convolution is generally applied to compute the dispersive transport of solutes from a time-varying source (e.g., Charbeneau 2000, 408, see also Carslaw and Jaeger 1959, 31).

Besbes and de Marsily (1984) adopted the gamma (Pierson III) function to represent impulse responses, which was based on solutions of the response of series of reservoirs as given by Nash (1959). They characterized it efficiently by its moments. Maas et al. (2006) expanded their method showing that these moments can be computed with a steady-state ground water model, thus allowing the modeling of the transient flow by convolution in a second step without the need of a transient model. This may open new ways for transient modeling with the analytic element method. Olsthoorn (2006) used convolution to interpret a multiwell pumping test in Egypt to deal effectively with many hundreds of pump cycles that occurred during the test. As will be shown, any standard (linear) numerical ground water model may be used to obtain the responses necessary to further analyze the ground water system by convolution.

Background

Convolution is a form of superposition. It uses the response of a system caused by an excitation, a pulse, to subsequently simulate the effect of arbitrary space or time-varying input.

1. Every physical system has an impulse response, R_I or, equivalently, a unit step response, R_S, which relates a given system input to its output (e.g., pumping to drawdown, rain to head fluctuation).
2. If the system is linear, then this impulse response is unique. This is equivalently true for the unit step response.
3. Either impulse response or unit step response contains all the dynamic information of the system necessary to relate its input to its output.

Figure 1 shows an impulse response, R_I. It is defined for an impulse, that is, an infinitely short excitation whose contents is fixed to unity, hence, $\lim_{\Delta\tau \to 0}(F\Delta\tau) = 1$. The dimensions of input and output generally differ. For instance, the head change may be the output of a rain shower as input. If the contents of the pulse $F\Delta\tau \neq 1$, then the impulse response is scaled (multiplied) by $F\Delta\tau$ to obtain the system response as indicated in Figure 1. In the case of a sequence of pulses, each one has its own scaled and horizontally shifted response. The results are added as shown in Figure 2. This is regular superposition in time presented in all textbooks on hydrogeology.

Convolution takes another perspective. It reverses the impulse response, R_I, relative to the time of interest, t, as shown by the continuous line in Figure 3 and, by so doing, aligns the values of R_I with the actual pulses $F(t - \tau_I)\Delta\tau$. This reduces the computation to an inner product between the pulses and their corresponding impulse response values. This is indicated in Figure 3. This process may also be seen as a moving average in which

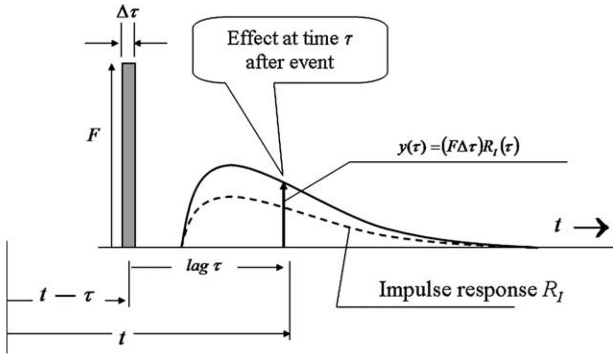

Figure 1. The impulse response (dashed line) is the reaction of the system due to an impulse such that $\lim_{\Delta t \to 0}(F\Delta\tau) = 1$. The system response $y(\tau)$ (continuous line) is the result of the actual pulse. It is obtained by scaling (multiplying) the impulse response by the magnitude of the actual pulse, $F\Delta\tau$.

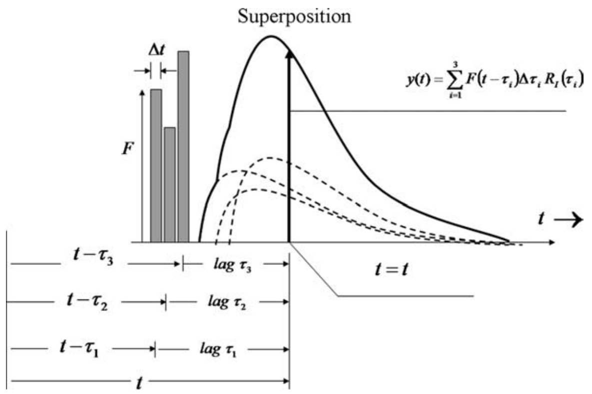

Figure 2. Superposition of three pulses. The dashed lines are the three scaled and shifted pulse responses. The continuous line is the overall response obtained by superposing the responses of the three individual pulses.

the pulses are weighed by an impulse response, which is moving along with t.

It may thus be expressed as follows:

$$s(t) = \sum_{i=0}^{\infty} (F(t - \tau_i)\Delta\tau_i)R_{\mathrm{I}}(\tau_i) \tag{1}$$

With $\mathrm{d}\tau \rightarrow 0$, this translates to Duhamel's (1797 to 1872) convolution integral for continuous time as follows:

$$s(t) = \int_{\tau=0}^{\infty} F(t - \tau)R_{\mathrm{I}}(\tau)\mathrm{d}\tau \tag{2}$$

In words, to obtain the system response s at a given time t that we are interested in, the input (excitation) at time τ earlier, that is, $F(t - \tau)$, is weighed with the corresponding impulse response value $R_{\mathrm{I}}(\tau)$ and summed (integrated).

A less intuitive often used equivalent form is $s(t) = \int_{\tau=0}^{\infty} F(\tau)R_{\mathrm{I}}(t - \tau)\mathrm{d}\tau$. Further, mathematicians mostly use $-\infty$ as the lower bound of the convolution integral; however, this is equivalent because the impulse response is always zero for $\tau < 0$.

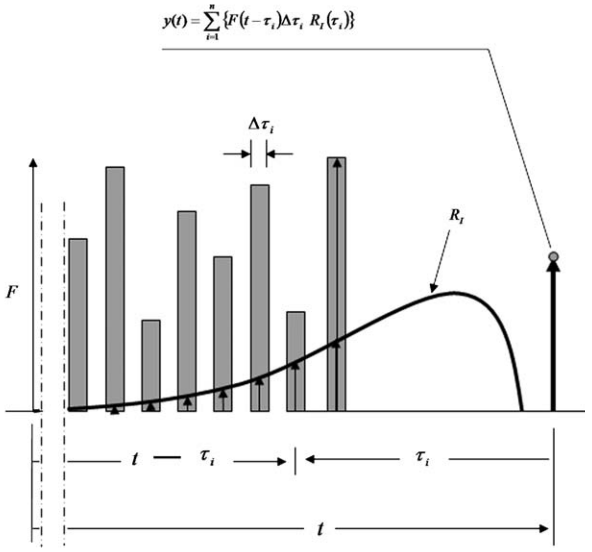

Figure 3. Convolution by reversing the impulse response relative to the time of interest, t, and subsequently multiplying the pulse magnitudes with the so-aligned impulse response values reduces the computation to the inner product $y(t)$.

One may apply Equation 1 to compute the convolution integral approximately, but it can only be exact in the limit $\Delta\tau \rightarrow 0$. In fact, Equation 1 may be inaccurate with impulse responses having a sharp peak at early times, as is the case with the well functions of Theis and Hantush. To be accurate, we prefer the exact solution for $F(t)$ piecewise constant, that is, one value $F_k = F(t_k \rightarrow t_{k+1})$ for each time step, with $-\infty < k < \infty$. When counting lag time τ and time steps $\Delta\tau$ by index i, backward in time, starting from the time of interest t_k, so that $0 \leq i < \infty$ (Figure 4), then the corresponding convolution integral becomes:

$$\begin{aligned} s_k = s(t_k) &= \sum_{i=1}^{\infty} F_{k-i} \int_{\tau_{i-1}}^{\tau_i} R_{\mathrm{I}}(\tau)\mathrm{d}\tau \\ &= \sum_{i=1}^{\infty} F_{k-i} R_{\mathrm{B}}(\tau_{i-1}, \tau_i) = \sum_{i=1}^{\infty} F_{k-i} R_{\mathrm{B},i} \end{aligned} \tag{3}$$

where $R_{\mathrm{B},i} = R_{\mathrm{B}}(\tau_{i-1}, \tau_i)$ is the so-called block response, that is, the effect of an excitation of unit strength during real-time step i.

Therefore, Equation 3 is exact for arbitrary time-step sizes and a piecewise constant excitation during each of them.

A continuous excitation starting at $\tau = 0$ yields the so-called step response $R_{\mathrm{S}}(\tau)$ as follows:

$$R_{\mathrm{S}}(\tau) = \int_0^{\tau} R_{\mathrm{I}}(\xi)\mathrm{d}\xi \quad \text{or equivalently}$$
$$R_{\mathrm{I}}(\tau) = \frac{\partial R_{\mathrm{S}}(\tau)}{\partial \tau} \tag{4}$$

Hence, the block response is given by:

$$R_{\mathrm{B}}(\tau_{i-1}, \tau_i) = R_{\mathrm{S}}(\tau_i) - R_{\mathrm{S}}(\tau_{i-1}) \quad \text{with } i > 0 \text{ and } R_{\mathrm{S},0} = 0 \tag{5}$$

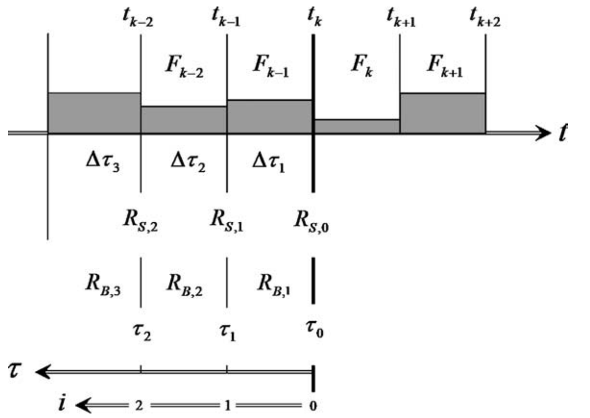

Figure 4. Counting when computing the convolution discretely for point $t = t_k$ using piecewise constant excitation F_k. Time t as well as index k run in principle form $-\infty$ to $+\infty$, while the lag time τ and index i from 0 to $+\infty$ as well as the step and block response values R_{S} and R_{B}. The location of the impulse response values R_{I} are not indicated because they are not uniquely defined in this discrete convolution scheme.

If we compute the step response for a series of times at which the value of the piecewise constant excitation changes, then the block response is just difference of the consecutive step responses.

The time step $\Delta\tau_i = \tau_i - \tau_{i-1}$ has arbitrary length; it does not have to be constant during a convolution.

Summarizing:

$$\begin{aligned} s_k = s(t_k) &= \sum_{i=1}^{\infty} F_{k-i} R_{B,i} = \sum_{i=1}^{\infty} F_{k-i}\left\{R_{S,i} - R_{S,i-1}\right\} \\ &= \int_0^{\infty} F(t-\tau)\frac{\partial R_S(\tau)}{\partial \tau}\mathrm{d}\tau = \int_0^{\infty} F(t-\tau) R_I(\tau)\mathrm{d}\tau \\ &\approx \sum_{i=1}^{\infty} F_{k-i} R_{I,i}\Delta\tau_i \end{aligned} \quad (6)$$

The last expression of Equation 6 is approximate and potentially inaccurate, while it depends on the location within the time step at which the impulse response $R_{I,i}$ is taken.

Clearly, the second expression in Equation 6 is the standard method of superposition in time as taught in ground water hydrology classes. F may be the extraction Q from a well and R_s the Theis well function divided by $4\pi T$.

Initial Condition

Convolution takes into account the excitation back into an infinite past (τ and i run to infinity in the equations). In practice, this past is limited to the available data, which may be less than what is required by the length of the impulse response. This is no problem when simulating pumping tests because the pumping before the test is assumed to always have been zero. So, we put as many zeros as we need in front of our pumping data. This is automatically the case in the spreadsheet example, where the cells below the excitation column are all empty, that is, zero. In general, however, if our impulse response is 15 years long, we also need 15 years of excitation data before the period of interest, as is the case in the numerical example that follows.

This looks like a disadvantage of convolution, because, at least theoretically, this is not necessary with a transient numerical model, where a run may start with the actual situation at an arbitrary time, without considering the past. However, in reality, we never know the true initial situation. Therefore, we usually start the model long before our period of interest, to make sure the simulation is consistent over the actual period of interest. Sometimes, we initialize the transient model with some steady state. But this is incorrect with models that are never in equilibrium, that is, with virtually all models. Therefore, this difference between a ground water model and convolution is merely theoretical without practical meaning.

If we have insufficient excitation data of the past, its impact may be mitigated by convolving with the excitation F minus its long-time average $\overline{F}$. Afterward, we add the result of convolution with the average excitation. Equivalently, we may put the necessary average excitation values in front of the available excitation data. Either method is equivalent to initializing a transient ground water model with a steady state at the start of the period of which we have actual data.

The length of the impulse response may be infinite. This is the case when no steady state is possible, as is the case with the Theis drawdown. As we know, the Theis drawdown can be finite only if the average extraction over an infinite past is zero. For a pumping test, this is always the case. The same is true for a pumping station in an infinite desert, but we must then convolve from the moment it first started pumping. This too is no different from the required length of the run of a transient ground water model.

Analytical Transient Solutions are Convolutions

Analytical solutions of transient ground water flow may be seen as convolutions. We may illustrate this on the hand of the Theis solution for the drawdown s[L] due to a well pumping at constant flow $Q[\mathrm{L}^3\mathrm{T}^{-1}]$ since $\tau = 0$ from an aquifer with storage coefficient S[-] and transmissivity $T[\mathrm{L}^2\mathrm{T}^{-1}]$.

$$s(\tau) = \frac{Q}{4\pi T} W(u) = \frac{Q}{4\pi T}\int_u^{\infty} \frac{\mathrm{e}^{-y}}{y}\mathrm{d}y \text{ with } u = \frac{r^2 S}{4T\tau} \quad (7)$$

This is equivalent to the following:

$$s(\tau) = Q\frac{W(u)}{4\pi T} = QR_s(\tau) \quad (8)$$

With R_s, the step response, and using Equation 5:

$$\frac{\partial s(\tau)}{\partial \tau} = Q(t-\tau)\frac{\partial R_s(\tau)}{\partial \tau} = Q(t-\tau)R_I(\tau) \quad (9)$$

With $R_I(\tau)$, the impulse response, we now use $Q(t-\tau)$ because Q is the value corresponding to the extraction at lag time τ before the time of interest, t; it no longer has to be a constant.

Hence, the convolution integral:

$$s(t) = \int_0^{\infty} Q(t-\tau)R_I(\tau)\mathrm{d}\tau \quad (10)$$

making it a function of time t only is valid for an arbitrary time-varying extraction.

In particular, for Theis:

$$R_I(\tau) = \frac{\partial R_S(\tau)}{\partial \tau} = \frac{1}{4\pi T}\frac{\mathrm{e}^{-u}}{\tau} \quad (11)$$

Appendix 1 demonstrates how convolution may be done in Excel. It implements the Theis solution for arbitrary varying extraction.

Generality and Effectiveness

Because convolution tackles arbitrary varying input straightforwardly, we may benefit from it in many practical situations. To give an example, one may use it to extract aquifer parameters from regular pumping station operations without organizing separate pumping tests. This may be done by combining pressure transducers in wells with the normal and automatically registered frequent pumping cycles on existing ground water pumping stations (Olsthoorn 2006).

The Excel implementation given in Appendix 1 is quite general because the analytical functions for the Theis case may be replaced by any other appropriate function for the situation at hand. Many transient analytic solutions such as those in Carslaw and Jaeger (1959) and Bruggeman (1999) may readily be represented in the form of convolution integrals.

In fact, it is immaterial by what method the required system responses are determined. It may be done analytically, like most of the referenced authors did, but also from a numerical model (e.g., Besbes and de Marsily 1984) or directly from field data through a time series analysis (e.g., Maas et al. 2006).

Convolution is effective because it allows using exact analytical solutions for simple situations to simulate arbitrarily varying input at any time resolution. It is also effective with numerical modeling because it allows simulating the effect of arbitrary varying input at arbitrary locations without rerunning a large and complex model repeatedly. Clearly, convolution is limited to neither recharge and heads nor pumping and drawdown. In fact, any input may be convolved with the appropriate impulse response to simulate any output of a linear model. Last but not least, convolution is effective because of its simple and straightforward implementation, even in a spreadsheet, as shown in Appendix 1.

Example Using a Numeric Model

In the sequel, convolution will be applied to a practical problem by exploiting a numerical model to obtain the unit step response of the water table due to rain recharge. We will simulate water table fluctuations in urban Heemstede, the Netherlands, in high time resolution without repeatedly rerunning the numerical model.

(To see the location, use Google Earth and type Heemstede, the Netherlands or 52°21′28″N, 4°37′05″E in the *Fly To* tab. Activate the Roads button on the *Layers* tab to see the two streets Van de Spiegellaan and Van Slingelandtlaan referenced in Figures 9 and 10).

The hydrologic objective is to separate distant from local causes of the ground water nuisance near the center of Heemstede (Heemstede Ctr in Figure 5) during the wet period from 1998 to 2002. It was stated in intensive ground water debates of the time that the seasonal water table fluctuations were due to the distant 4-km-wide coastal dune area rather than local shortage of the urban drainage capacity during the four wet years.

The aim here is to show (1) that we can accurately simulate the measured water table time series in the urban area by convolution using the unit step response obtained from an existing numerical model and (2) that we may use this convolution to separate the influence of the local more rapidly drained ground water system from that of the distant dune zone, which is much more slowly drained.

Figure 5 gives a cross-section ground water model with steady-state head and streamlines. The flow direction is indicated by the arrows. The model is 20 km long (Figure 6) and perpendicular to the coast. Its end

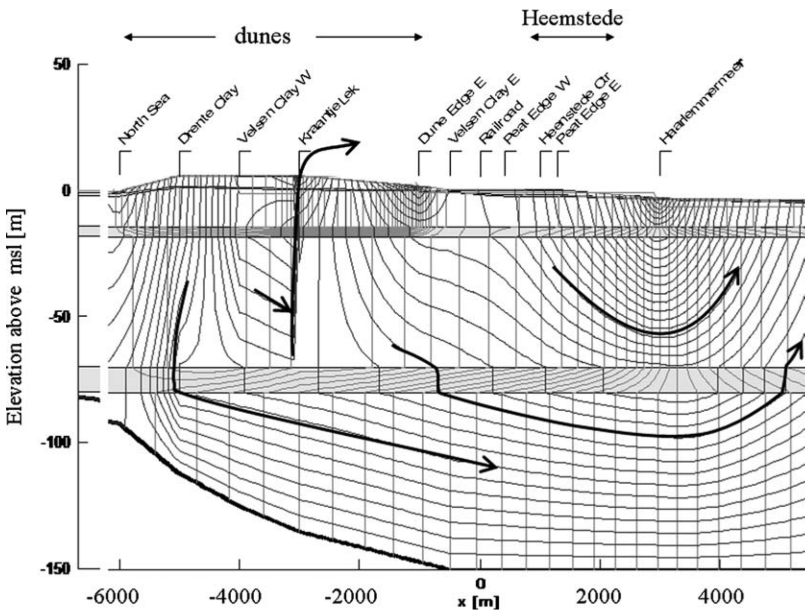

Figure 5. Cross-section model perpendicular to the Dutch coast through the city of Haarlem, the Netherlands. It crosses the 4- to 5-km-wide coastal dune zone and traverses the urban area of Heemstede into the Haarlemmermeer polder, a former lake put dry in 1852 with surface water levels at 6 m below mean sea level. The entire section is 20 km long; its Google Earth end coordinates are given in Figure 6 and in the text. Steady-state head and streamlines are shown; the direction of flow is indicated by the arrows.

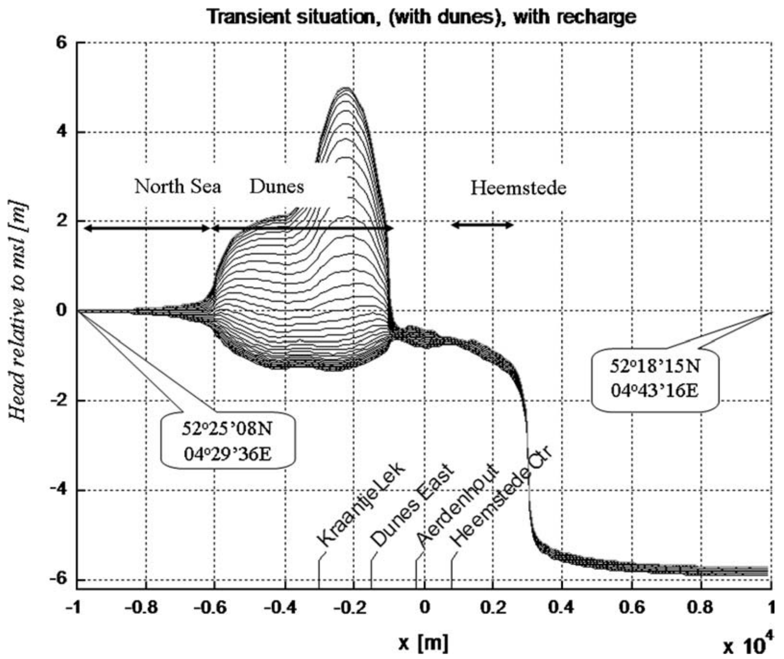

Figure 6. The 20-km cross section showing the climbing water table due to continuous recharge of 1 mm/d starting from a steady-state zero-recharge initial situation. This 6000 d simulation time was needed to reach equilibrium throughout the model. This change of head over time is the unit step response. The simulation was done with 60 exponentially increasing time steps.

coordinates are (52°25′08″N, 44°29′36″E) and (52°18′15″N, 4°23′16″E).

The dunes form a large unsaturated 4- to 5-km-wide zone parallel to the North Sea coast, virtually without drainage. Farther inland, more intensively drained urban areas are present, like the villages Aerdenhout and Heemstede (locations shown in Figure 6). The eastern part of the system is dominated by the deep polder Haarlemmermeer. This is a previous lake that was put dry in 1852. The surface water level in this polder is maintained at 6 m below mean sea level, which causes a strong upward seepage. The resistance of the aquitards in the cross section varies according to the presence of clay and peat layers.

The slowness of the large dune system causes accumulation of ground water over consecutive wet years, as was the case in 1988 to 1989, 1994 to 1995, and, especially, 1998 to 2002 (Figure 8, curves "Kraantje Lek" and "Dunes East"). This multiyear accumulation does not occur in the urban area due to more intensive drainage to local surface water (Figure 8, curves Aerdenhout and Heemstede). Nevertheless, problems arose in the wet period 1998 to 2002, such as wet cellars and submerged gardens over extended periods.

The necessary simulations for this analysis may be done with any transient numerical model. However, this is mostly cumbersome and time consuming, especially when a high time resolution is desired and the model is large. Convolution may be a more effective method once the required step responses have been obtained from the available numerical ground water model.

To compute the unit step response of the (rain) recharge, the model was run with constant unit recharge from a steady-state zero-recharge solution. The simulation was 15 years to achieve equilibrium. This computation requires a single and simple stress period with minimal input. Figure 6 shows the positions of the water table for consecutive times starting from the zero-recharge equilibrium situation. The number of time steps may be limited by exponentially increasing their size as was done here, but this requires a subsequent interpolation to obtain the desired linear time scale necessary for the convolution. (I used 60 steps increasing from 0.1 to 1020 d and subsequently interpolated to daily steps.)

The unit step response was finally computed by subtracting the steady-state initial solution and, if necessary, dividing by the applied constant recharge. The unit step response is then available for every point of the model, including specific points of interest (Figure 7), as well as for any other output of the model such as discharges. This analysis requires that the model behave linearly during the simulation.

By taking the difference of subsequent values of the unit step response, we obtain the block responses for the selected time step as in Equation 5.

Finally, the convolution was carried out with these block responses R_B and the actual rain recharge, n:

$$s_k = s(t_k) = \sum_{i=1}^{\infty} n_{k-i} R_{B,i} \tag{12}$$

This convolution may be done in Excel, Maple, or any other suitable program. I used the function *filter*(..) in Matlab.

The multiyear convolution results with a daily resolution are shown in Figure 8 for four selected points in

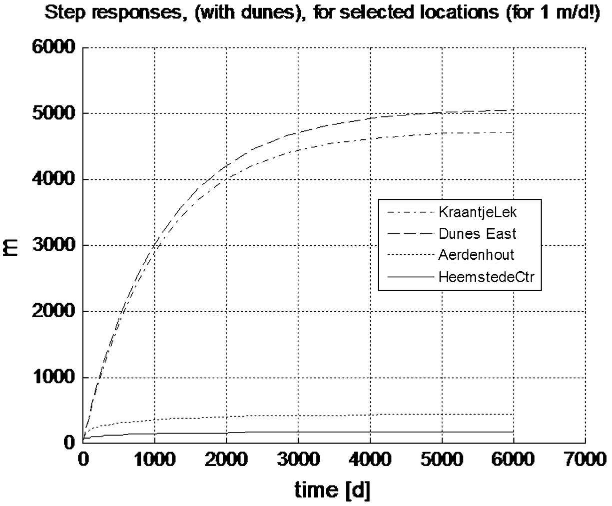

Figure 7. Unit step responses obtained from the numerical model for the locations indicated in Figure 6.

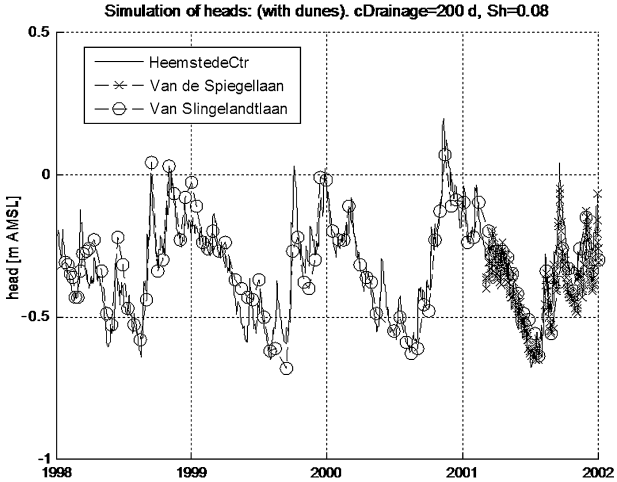

Figure 9. Simulated water table for Heemstede Center vs. measured values in the period 1998 and 2002 in two observation wells in urban Heemstede. The computed line for Heemstede Center is a detail of the corresponding line in Figure 8.

the shallow aquifer of the model (Figure 6 for the locations).

The simulation used the available recharge data from 1950. The period 1950 to the start of the curves in Figure 8, that is, 1972, is long enough to initialize the convolution, which has a 16-year-long impulse response.

The accumulating effect of the large dune system over multiyear wet periods is obvious from this simulation, as is the much smaller effect in the urban area.

The convolution results may readily be compared with the measurements. Figure 9 shows good correspondence between the simulated and measured data for two observation wells in Heemstede.

To answer the question regarding the impact of the dune system on the urban water table, the recharge in the urban area was set to zero in the ground water model, which was then rerun to compute a new set of unit step responses. The simulated urban water table fluctuation is then caused by the distant dune system alone. The convolution was redone yielding the results in Figure 10. Comparison of Figures 9 and 10 makes clear that there is indeed an influence of the dune system on the shallow urban ground water but that it is small. In conclusion, the fluctuations in the village are dominated by the local ground water system.

Conclusions

Convolution is an effective and efficient method to compute the dynamic response of linear systems that are subjected to arbitrary stresses; this includes many ground water systems. Broader consideration for its use in the field of ground water hydrology was already advised by

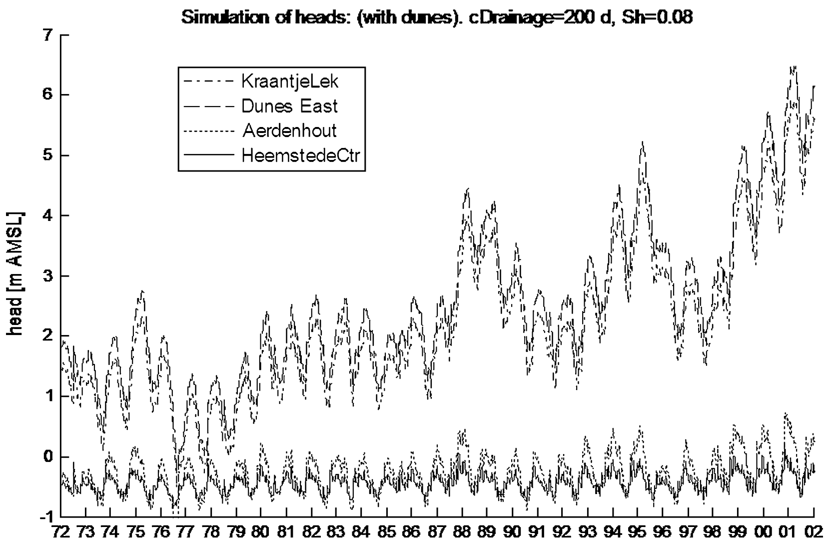

Figure 8. Simulated water table for the selected locations by convolution using the numerically obtained unit step responses in Figure 7 and the actual rain recharge. The climbing water table in the dune points "Kraantje Lek" and "Dunes East" during the wet periods 1988 to 1989, 1994 to 1995, and, especially, 1998 to 2002 is remarkable.

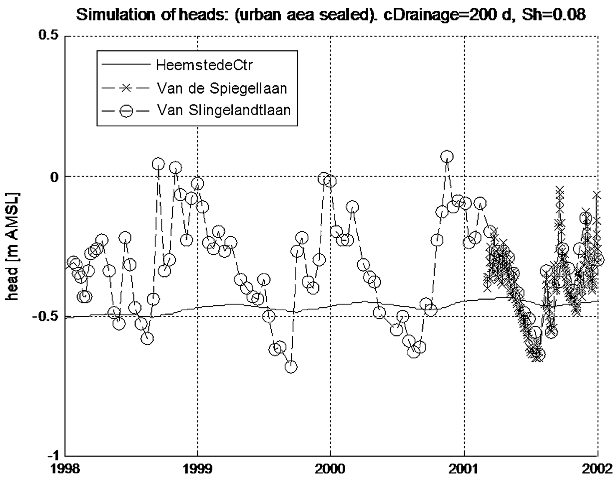

Figure 10. Same as Figure 9 but with recharge in the urban area set to zero. This way, the simulated curve for Heemstede Center quantifies only the contribution of the distant dune zone to the local water table variation.

Hall and Moench (1972), but unfortunately, it is still missing in most textbooks in our field, which limits a widespread practical application.

Most ground water applications of convolution concern the interaction between surface and ground water, but its potential is much wider, as some papers show. The current paper demonstrates the use of an existing ground water model to compute the unit step response followed by convolution to analyze detailed dynamic water table fluctuations without further use of the model. Another example is aquifer analysis by exploiting ground water dynamics around ground water pumping stations with frequent or random pump cycles (Olsthoorn 2006). This might replace dedicated pumping tests.

In general, any analytic formula providing the required system response, such as the Theis and Hantush well functions, can be used to analyze complex highly dynamic systems by convolution. But the required system response may also be obtained by other methods. As was illustrated, one of them is by running an existing ground water model for a single and simple stress period with constant input from a steady-state situation until equilibrium has been reached.

Convolution may readily be done in existing mathematical programs such as Maple and Matlab. But a spreadsheet may also suit one's needs, as was shown in Appendix 1.

There are probably thousands of other ground water–related applications that would benefit from convolution. Therefore, I suggest including convolution in all general ground water courses and textbooks to facilitate a more widespread application of this valuable method.

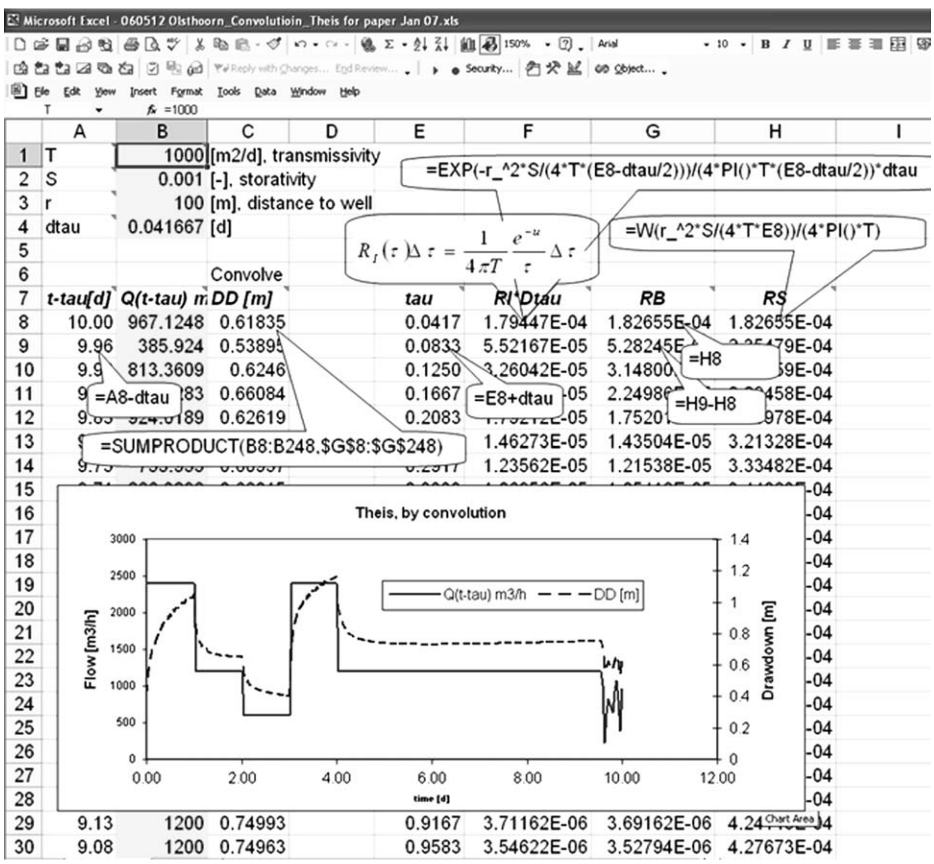

Figure 11. Example spreadsheet computing by convolution the drawdown due to time-varying extraction. All formulas are shown. Names *T*, *S*, *r_*, and *dtau* were given to the cells B1:B4. The last *Q* values (B8:B18) were random, which causes the violent variation at the end of the graph.

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References

Abramowitz, M., and I. Stegun. 1972. *Handbook of Mathematical Functions*. New York: Dover Publications.

Barlow, P.M., and A.F. Moench. 1998. Analytical solutions and computer programs for hydraulic interaction of stream-aquifer systems. USGS Open-File Report 98–415A. Reston, Virginia: USGS.

Barlow, P.M., L.A. DeSimone, and A.F. Moench. 2000. Aquifer response to stream-stage and recharge variations. II. Convolution method and applications. *Journal of Hydrology* 230, no. 3–4: 211–229.

Bear, J. 1979. *Hydraulics of Groundwater*. New York: McGraw Hill.

Bear, J. 1972. *Dynamics of Fluids in Porous Media*. New York: Dover.

Besbes, M., and G. de Marsily. 1984. From infiltration to recharge: Use of parametric transfer function. *Journal of Hydrology* 74, no. 3–4: 271–293.

Bruggeman, G.A. 1999. *Analytical Solutions of Geohydrological Problems*. Amsterdam: Elsevier.

Carslaw H.S., and J.C. Jaeger. 1959. *Conduction of Heat in Solids*. Oxford, U.K.: Oxford University Press.

Charbeneau, R.J. 2000. *Ground water Hydraulics and Pollutant Transport*. ISBN 0-13-975616-7. Upper Saddle River, New Jersey: Prentice Hall.

Cooper H.H., and M.J. Rorabaugh. 1963. Ground water movements and bank storage due to flood stages in surface streams. USGS Water Supply Paper 1536-J. Reston, Virginia: USGS.

de Marsily, G. 1986. *Quantitative Hydrogeology.* ISBN 0-12-208915-4/0-12-208-916-2. Orlando, Florida: Academic Press.

DeSimone, L.A., and P.M. Barlow. 1999. Use of computer programs STLK1 and STWT1 for analysis of stream-aquifer hydraulic interaction: USGS Water-Resources Investigations Report 98-4212. Reston, Virginia: USGS.

Dingman, S.L. 2002. *Physical Hydrology.* ISBN 0-13-099695-5. Englewood Cliffs, New Jersey: Prentice Hall.

Domenico, P.A., and F.W. Schwartz. 1998. *Physical and Chemical Hydrogeology.* New York: John Wiley and Sons Inc.

Fetter, C.W. 2001. *Applied Hydrogeology.* Upper Saddle River, New Jersey: Prentice Hall.

Fitts, C.R. 2002. *Ground Water Science*. Amsterdam, The Netherlands: Academic Press.

Freeze, R.A., and J.A. Cherry. 1979. *Ground Water.* Englewood Cliffs, New Jersey: Prentice Hall.

Hall, F.R., and A.F. Moench. 1972. Application of the convolution equation to stream-aquifer relationships. *Water Resources Research* 8, no. 2: 487–493.

Illangasekare, T.H., and H.J. Morel-Seytoux. 1986. A discrete kernel simulation model for conjunctive management of a stream-aquifer system. *Journal of Hydrology* 86, no. 3–4: 319–338.

Knight, J.H., M. Gilfedder, and G.R. Walker. 2004. Impacts of irrigation and dryland development on ground water discharge to rivers—A unit response approach to cumulative impacts analysis. *Journal of Hydrology* 303, no. 1–3: 79–91.

Kruseman, G., and N.A. de Ridder. 1994. *Analysis and Evaluation of Pumping Test Data*. Bulletin 11, ILRI publication 47, Completely revised 2nd ed. ISBN 90-70745-207. Wageningen, The Netherlands: ILRI.

Kruseman, G.P., and N.A. de Ridder. 1970. *Analysis of Pumping Test Data*. Wageningen, The Netherlands: ILRI.

Lee, T.-C. 1999. *Applied Mathematics in Hydrogeology.* Boca Raton, Florida: CRC Press LCC.

Maas, C., J. von Asmuth, and M. Bakker. 2006. Natural ground water dynamics 2: Time series analysis. In Proceedings Modflow and More, *Managing Ground Water Systems, Golden, Co, 21-24 May*, vol. II, 752–754. Golden, Colorado: International Ground Water Modeling Center (GWMC), Colorado School of Mines.

Moench, A.F., and P.M. Barlow. 2000. Aquifer response to stream-stage and recharge variations. I. Analytical step-response functions. *Journal of Hydrology* 230, no. 3–4: 192–210.

Moench A.F., and C.C. Kisiel. 1970. Application of the convolution relation to estimating recharge from an ephemeral stream. *Water Resources Research* 6, no. 4: 1087–1094.

Nash, J.E. 1959. Systematic determination of unit hydrograph parameters. *Journal of Geophysical Research* 64, no. 1: 111–115.

Olsthoorn, T.N. 2006. A pumping test with thousands of pump switchings in just a couple of days. In *Proceedings of the International Conference on Analytic Element Modeling. Manhattan, KS, 14–17 May 2006*, 208–219. Manhattan, Kansas: Kansas State University.

Oppenheim, A.V., A.S. Willsky, and S.H. Nawab. 1997. *Signals and Systems*, 2nd ed. ISBN 0-13-651175-9. London: Prentice Hall.

Pinder, G.F., J.D. Bredehoeft, and H.H. Cooper. 1969. Determination of aquifer diffusivity from aquifer response to fluctuations in river stage. *Water Resources Research* 5, no. 4: 850–855.

Rassam, D., G. Walker, and J. Knight. 2004. Applicability of the unit response equation to assess salinity impacts of irrigation development in the Mallee Region. CSIRO Land and Water Technical Report No 35/04. Adelaide, Australia: CSIRO.

Rowe, P.P. 1960. An equation for estimating transmissibility and coefficient of storage from river-level fluctuations. *Journal of Geophysical Research* 65, no. 10: 3419–2424.

Schwartz, F.W., and H. Zhang. 2003. *Fundamentals of Ground Water.* New York: John Wiley and Sons Inc.

Singh V.S., and C.P. Gupta. 1986. Hydrogeological parameter estimation from pump tests on lartge diameter wells. *Journal of Hydrology* 87, no. 3–4: 223–232.

Strack, O.D.L. 1989. *Groundwater Mechanics.* ISBN 0-13-365412-5. Englewood Cliffs, New Jersey: Prentice Hall.

Todd, D.K., and L.W. Mays. 2005. *Groundwater Hydrology.* 3rd ed. ISBN 0-471-05937-4. Hoboken, New Jersey: John Wiley and Sons Inc.

Venetis, C. 1970. Finite aquifers: Characteristic responses and applications. *Journal of Hydrology* 12, no. 1–2: 53–62.

Venetis, C. 1968. On the impulse of an aquifer. *Bulletin of the International Association of Scientific Hydrology* 13, 36–139.

Zhang, W.Z. 1992. Transient ground water flow in an aquifer-aquitard system in response to water level changes in river or canals. *Journal of Hydrology* 133, no. 3–4: 233–257.

Appendix 1

Convolution in Excel (Theis with Arbitrary Input)

Figure 11 shows an Excel spreadsheet that computes the drawdown for an arbitrary varying well extraction. For this, the Theis well function is needed, which be implemented as explained in Appendix 2.

Cells B1:B4 of this spreadsheet contain transmissivity *T*, storativity *S*, the distance *r* from the well for which the drawdown is to be computed, and the chosen time step *dtau* (in this figure, 1/24 d = 1 h).

Range A8 downward contains the actual time (d) sorted backward, so that the most recent time is at the top. Range B8 downward is the extraction (here in m^3/d) during each time step. Range C8 downward is the computed drawdown, that is, the convolution result as explained subsequently.

Range E8 downward is the lag time *tau* (same time steps as the extraction data but increasing downward).

Range H8 downward is the step response R_S. This is the Theis well function divided by ($4\pi T$) as in Equation 8.

Range G8 downward is the block response R_B computed as the difference of two consecutive step responses. It is the exact result of switching on the pump at $tau = 0$ and switching it off again at the end of the first time step, $tau = dtau$ (Equation 5).

Range F8 downward is the approximate block response computed as the impulse response R_I times the step size. Notice that the first few values differ from the exact block responses given in range G8 downward. This difference makes convolution results using range F8 downward inaccurate. However, convolution with the block responses in range G8 downward yields exact results.

The drawdown in range C8 downward is obtained by convolution, that is, by multiplying the block response with the extraction reversed in time (i.e., range B8 downward) and shifting by *tau*.

This range multiplication is done with the standard spreadsheet function = sumproduct(*range_a*,*range_b*) shown in cell C8, where *range_b* is the block response and *range_a* the varying extraction.

The formula is then copied downward to obtain the drawdown at other times. To obtain the correct shifting, $Q(t\text{-}tau)$, the block response, *range_b*, must be kept fixed when copying (by using $s in the range in the formula, e.g., $G$8:$G$248) and the extraction range must not be kept fixed, for example, B8:B248. The value 248 corresponds to the length of the data in this particular spreadsheet, which was 248 h of pumping. The length of the pumping data range should be taken in the two ranges of the sumproduct-function to prevent missing part of the tail of the block response. This assumes that no pumping occurred before the first available pumping data.

The last step is to draw a graph of the drawdown vs. time. To this end, select the range to be plotted and choose the scatter plot chart type. To include Q vs. time in the same graph (but on a different axis), click on a curve in the chart, then right click, choose "format data series," choose the Axis tab, and finally click "Secondary Axis."

In this sheet, arbitrary extraction values can be used in the Q range (B8 downward). The results are immediately reflected in the graph. The graph in Figure 11 shows that the extraction was piecewise constant over much of the extraction period and random over the past 10 h (see Q values in B8:B18).

To compare the sheet with the standard Theis solution, use a constant extraction in the entire Q range (B8:B248).

Appendix 2

Theis' Well Function (Exponential Integral)

Theis's well function $W(u)$, that is, the exponential integral $E_1(u)$, is tabulated in many ground water and mathematical textbooks (e.g., Abramowitz and Stegun 1972, equations 5.1.51 and 5.1.63). It is also standard available in many mathematical programs, like the function *expint*(-) in Matlab. It is missing in Excel, so we must implement it. This may be done using the well-known power series:

$$W(u) = -0.57721 - \ln(u) + u - \frac{u^2}{2 \times 2!} + \frac{u^3}{3 \times 3!} - \frac{u^4}{4 \times 4!} + \ldots \quad (13)$$

Implementing a new function in Excel is straightforward using Visual Basic (VB). In Excel, select VB from the toolbars that appear when right clicking the gray menu area on top of the screen. Then switch to VB by clicking the VB Editor icon on the VB toolbar. In VB, select from the main menu insert>module to create a place to store the function. Type the function as given subsequently. Switch back to Excel by clicking on the Excel icon (the top left of the VB screen). Then use the new function like any standard Excel function, for example, type = $W(3.2)$ in a cell, which should yield the value 0.01013.

```
Function W (u As Double) As Double
Dim n AS Integer, term As Double
W = 0
If u < 15 Then
n = 1
term = -u
W = -0.5772156649 - Log(u) - term
Do While Abs(term) > 0.000000001
term = -u * n/(n + 1) ^ 2 * term
W = W- term
n = n + 1
Loop
End If
End Function
```

$W(u)$ Directly in Excel

The power series:

$$W(u) = -0.57721 - \ln(u) - \sum_{i=1}^{\infty} \frac{(-1)^i}{i \times i!} u^i \quad (14)$$

may also be implemented using the worksheet function SERIESSUM:

$$-0.57721566 - \ln(u) - \text{SERIESSUM}(u, 1, 1, \text{coefficients}),$$

where coefficients is the name of the range with the coefficients $\frac{(-1)^i}{i \times i!}$

with 45 coefficients one may use

$$\text{if}(u > 15, 0, -0.57721566 - \ln(u) - \text{SERIESSUM}(u, 1, 1, \text{coefficients}))$$