

# MULTISKYRMIONS AND BARYONIC BAGS.

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Analytical treatment of skyrmions given by rational map ( $RM$ ) ansaetze proposed recently for the Skyrme model is extended for the model including the 6-th order term in chiral field derivatives in the lagrangian and used for the calculations of different properties of multiskyrmions. At large baryon numbers the approximate solutions obtained are similar to the domain wall, or to spherical bubbles with energy and baryon number density concentrated at their boundary. Rigorous upper bound is obtained for the masses of  $RM$  multiskyrmions which is close to the known masses, especially at large  $B$ . For the 6-th order variant the lower bound for masses of  $RM$  skyrmions is obtained as well. The main properties of the bubbles of matter are obtained for arbitrary number of flavours. They are qualitatively the same for the 4-th and 6-th order terms present in the lagrangian, although differ in some details.

## 1 Introduction

Soliton models of different kinds are used in various fields of physics. In elementary particle physics the soliton models provide a concept of baryons as extended in space objects, opposite to the concept of the point-like objects, usual for the quantum field theory. The chiral soliton approach, starting with several basic principles and ingredients incorporated in the model lagrangian [1, 2] provides realistic and even satisfactory description of baryons and baryonic systems. The latter are obtained within this approach as quantized solitonic solutions of equations of motion, characterized by the so called winding number or topological charge which is identified with the baryon number  $B$ . Numerical studies have shown that the chiral field configurations of lowest energy possess different topological properties - the shape of the mass and  $B$ -number distribution - for different values of  $B$ . It is a sphere for  $B = 1$  hedgehog [1], a torus for  $B = 2$ , tetrahedron for  $B = 3$ , cube for  $B = 4$ , and higher polyhedrons for greater baryon numbers. The symmetries of various configurations for  $B$  up to 22 and their masses have been determined in [3] (the references to earlier original papers where the symmetries of configurations with smaller baryon numbers have been determined can be found in [3, 4]). These configurations have one-shell structure and for  $B > 6$  all of them, except two cases, are formed from 12 pentagons and  $2B - 14$  hexagons; in carbon chemistry similar structures are known as fullerenes [3]. The mass and baryon number densities for these configurations are concentrated along the edges of polyhedrons. All these configurations can be made of  $2B - 2$  slightly deformed torus-like configurations glued together, which can be considered by this reason as elementary building blocks for multiskyrmions. As will be shown here, the dimensions of these elementary cells do not depend on  $B$  when  $B$  is large enough.

The so called rational map ( $RM$ ) ansatz, proposed for the  $SU(2)$  skyrmions in [5] and widely used now, in present paper as well, allows to simplify the problem of finding the configurations of lowest energy. For the  $RM$  ansatz the minimization of the skyrmions energy functional proceeds in two steps: at first step the map from  $S^2 \rightarrow S^2$  is minimized for the  $SU(2)$  model (for the  $SU(N)$  model it is a map from  $S^2 \rightarrow CP^{N-1}$ , [6]), and, second, the energy functional depending on skyrmion profile as a function of distance from center of skyrmion is minimized. As will be shown here, just the second step can be done analytically with quite good accuracy. Many important properties of the  $RM$  multiskyrmions can be

studied in this way, and some of them do not depend on result of the first step. This allows to make certain conclusions for the arbitrary large  $B$  and for any number of flavours  $N_F = N$  independently of presence of numerical calculations. Without difficulties the consideration has been extended to the variant of the model with the 6-th order terms in chiral derivatives included into lagrangian (the SK6 variant of the Skyrme model). Remarkably, that for the SK6 variant of the model the dependence of the results on the first step of calculation is even weaker than for the SK4 variant.

Beginning with [1, 2], the chiral soliton models have been considered as a special class of models for baryons. Their connection with other models could be instructive and useful, and this is also an issue of present paper. In particular, it is shown that Skyrme-type models provide field theoretical realization of the bag model of special kind for baryonic systems.

## 2 Large $B$ multiskyrmions as spherical bubbles or domain walls

Here we consider the multiskyrmions in the general  $SU(N)$  case; detailed comparison of analytical results with numerical calculations is made in the  $SU(2)$  model and also in the  $SU(3)$  variant using the projector ansatz [6]. In the  $SU(2)$  model the chiral fields are functions of the profile  $f$  and the unit vector  $\vec{n}$ , according to definition of the unitary matrix  $U \in SU(2)$   $U = c_f + is_f \vec{n} \vec{\tau}$ . For the ansatz based on the rational maps the profile  $f$  depends only on the variable  $r$ , and the components of vector  $\vec{n}$  - on angular variables  $\theta, \phi$ .  $n_1 = (2 \operatorname{Re} R)/(1 + |R|^2)$ ,  $n_2 = (2 \operatorname{Im} R)/(1 + |R|^2)$ ,  $n_3 = (1 - |R|^2)/(1 + |R|^2)$ , where  $R$  is a rational function of variable  $z = tg(\theta/2)exp(i\phi)$  defining the map of degree  $\mathcal{N}$  from  $S^2 \rightarrow S^2$ .

The notations are used [5]

$$\mathcal{N} = \frac{1}{8\pi} \int r^2 (\partial_i \vec{n})^2 d\Omega = \frac{1}{4\pi} \int \frac{2idRd\bar{R}}{(1 + |R|^2)^2}$$

$$\mathcal{I} = \frac{1}{4\pi} \int r^4 \frac{[\vec{\partial} n_1 \vec{\partial} n_2]^2}{n_3^2} d\Omega = \frac{1}{4\pi} \int \left( \frac{(1 + |z|^2) |dR|}{(1 + |R|^2) |dz|} \right)^4 \frac{2idzd\bar{z}}{(1 + |z|^2)^2}, \quad (1)$$

where  $\Omega$  is a spherical angle. For  $B = 1$  hedgehog  $\mathcal{N} = \mathcal{I} = 1$ .  $\mathcal{N} = B$  for configurations of lowest energy.

For more general  $SU(N)$  case and using projector ansatz one obtains [6]

$$\mathcal{N} = \frac{i}{2\pi} \int dzd\bar{z} Tr[\partial_z P \partial_{\bar{z}} P],$$

$$\mathcal{I} = \frac{i}{4\pi} \int dzd\bar{z} (1 + |z|^2)^2 Tr[\partial_z P, \partial_{\bar{z}} P]^2, \quad (2)$$

$P$  is a projector,  $N \times N$  hermitian matrix,  $P = f f^\dagger / f^\dagger f$ ,  $Tr P = 1$ . For  $SU(2)$  case the 2-component column  $f = (R, 1)^T$ , and (1) can be obtained easily from formulas (2).

The classical mass of skyrmion for  $RM$  ansatz in universal units  $3\pi^2 F_\pi / e$  is [6, 7]:

$$M = \frac{1}{3\pi} \int \left\{ A_N r^2 f'^2 + 2B s_f^2 [1 + (1 - \lambda) f'^2] + (1 - \lambda) \mathcal{I} \frac{s_f^4}{r^2} + \lambda \mathcal{I} \frac{s_f^4}{r^2} f'^2 \right\} dr, \quad (3)$$

$r$  measured in units  $2/(F_\pi e)$ , the coefficient  $A_N = 2(N - 1)/N$  for symmetry group  $SU(N)$  [6], and this generalization provides a possibility to consider models with arbitrary number of

flavours  $N = N_F$  - essentially nonembeddings of  $SU(2)$  in  $SU(N)$ .  $\lambda$  defines the weight of the 6-th order term. If  $\lambda = 0$  we obtain the original Skyrme model variant (we call it the SK4-variant),  $\lambda = 1$  corresponds to the pure SK6-variant. The expression (3) for mixed variant was obtained in [7], where the properties of multiskyrmions with  $B \leq 5$  have been investigated in the modification of the Skyrme model with 6 - th order term included into lagrangian. For  $B = 2$  the results obtained in [7] agree with those obtained in [8] which correspond to the value of  $\lambda = 1$ . Besides axially symmetrical configurations with  $B \leq 5$  in the SK4 variant the torus-like configuration with  $B = 2$  was obtained in [8] also in the SK6 variant. For reasonable values of model parameters the binding energy was about twice greater than that for pure Skyrme model: it was found  $\sim 150 Mev$  instead of  $74 Mev$  for the variant of the SK4 model which provides fit for the masses of nucleon and  $\Delta$ -isobar.

It was important in deriving (3) that the integral over angular variables for the trace of the third power of  $[\partial_z P, \partial_{\bar{z}} P]$  equals to zero [7]:

$$\mathcal{M} = \frac{i}{8\pi} \int dz d\bar{z} (1 + |z|^2)^4 \text{Tr} [\partial_z P \partial_{\bar{z}} P]^3 = 0,$$

therefore, the structure (3) is specific for the rational map ansatz.

To find the minimal energy configuration at fixed  $\mathcal{N} = B$  one minimizes  $\mathcal{I}$ , and then finds the profile  $f(r)$  by minimizing energy (3). The inequality takes place:  $\mathcal{I} \geq B^2$  [5, 6]. The important consequence of (3) is that the symmetries of multiskyrmions in the SK6 variant [7] are the same as for the SK4 variant, because the quantity  $\mathcal{I}$  is the same for both variants.

Numerical calculations performed in [3]-[6] have shown, and the analytical treatment of [9] and here supports, that at large  $B$  and, hence, large  $\mathcal{I}$  the multiskyrmion looks like a spherical bubble with profile equal to  $f = \pi$  inside and  $f = 0$  outside. The energy and  $B$ -number density of this configuration is concentrated at its shell, similar to the domain walls system considered in [10] in connection with cosmological problems.

The lower bound for the mass of solitons can be obtained from (3) for the SK6 variant, similar to the SK4 variant, known previously. Using evident relations  $A_N f'^2 r^2 + \mathcal{I} s_f^4 f'^2 / r^2 \geq 2\sqrt{A_N \mathcal{I}} f'^2 s_f^2$ , and  $f'^2 \sqrt{A_N \mathcal{I}} + B \geq -2f'(B\sqrt{A_N \mathcal{I}})^{1/2}$  (recall that  $f' < 0$ ) we obtain from (3) for arbitrary value of real positive parameter  $\lambda \leq 1$ :

$$\frac{M(B, \lambda)}{B} \geq \frac{2\lambda}{3} \left( \frac{A_N \mathcal{I}}{B^2} \right)^{1/4} + \frac{1-\lambda}{3} \left[ 2 + \left( \frac{A_N \mathcal{I}}{B^2} \right)^{1/2} \right]. \quad (4a)$$

For  $\lambda = 0$  the known results of [5, 6] are reproduced, for the SK6 variant ( $\lambda = 1$ ) it is new bound. When the last term in (3) can be neglected, another lower bound can be obtained, see also [7]:

$$\frac{M(B, \lambda)}{B} \geq \frac{\sqrt{1-\lambda}}{3} \left[ 2 + \left( \frac{A_N \mathcal{I}}{B^2} \right)^{1/2} \right], \quad (4b)$$

which does not provide real restriction when  $\lambda$  is close to 1. Recall that the bound  $M/B > 1$  was obtained for arbitrary, not only  $RM$ ,  $SU(2)$  skyrmions first by Skyrme in the SK4 variant of the model [1].

Denote  $\phi = \cos f$ , then the energy (3) can be presented as

$$M = \frac{1}{3\pi} \int \left\{ \frac{1}{(1-\phi^2)} [A_N r^2 \phi'^2 + 2B(1-\phi^2)^2] + (1-\lambda) [2B\phi'^2 + \mathcal{I}(1-\phi^2)^2/r^2] + \lambda \mathcal{I}(1-\phi^2)\phi'^2/r^2 \right\} dr, \quad (5)$$

with  $\phi$  changing from  $-1$  at  $r = 0$  to  $1$  at  $r \rightarrow \infty$ . The first part of (5) is the second order term contribution into the mass, the second - the Skyrme term contribution, and the last, proportional to  $\lambda$ , - the 6-th order term. At fixed  $r = r_0$  the 4-th order term is proportional exactly to 1-dimensional domain wall energy. It is possible to write the second order contribution in (5) in the form:

$$M^{(2)} = \frac{1}{3\pi} \int \left\{ \frac{A_N r^2}{(1-\phi^2)} \left[ \phi' - \sqrt{\frac{2B}{A_N}} (1-\phi^2)/r \right]^2 + 2r \sqrt{2A_N B} \phi' \right\} dr,$$

and similar for the 4-th order Skyrme term. The equality  $\phi' = \sqrt{2B/A_N} (1-\phi^2)/r$  eliminates considerable part of integrand for  $M^{(2)}$ . Therefore, it is natural to consider function  $\phi$  satisfying the following differential equation:

$$\phi' = \frac{b}{2r} (1-\phi^2) \quad (6)$$

with constant power  $b$ , which has solution satisfying boundary conditions  $\phi(0) = -1$  and  $\phi(\infty) = 1$ :

$$\phi(r, r_0, b) = \frac{(r/r_0)^b - 1}{(r/r_0)^b + 1} \quad (7)$$

with arbitrary  $r_0$  - the distance from the origin of the point where  $\phi = 0$  and profile  $f = \pi/2$ .  $r_0$  can be considered as a radius of multiskyrmion, both  $b$  and  $r_0$  will be defined further by means of the mass minimization procedure. The radii of distributions of baryon number and mass in the soliton are close to  $r_0$ , see Section 5 below.

After substitution of this ansatz one obtains the soliton mass in the form:

$$M(B, b) = \frac{1}{3\pi} \int \left\{ (A_N b^2/4 + 2B)(1-\phi^2) + (1-\lambda)(Bb^2/2 + \mathcal{I})(1-\phi^2)^2/r^2 + \lambda \mathcal{I} b^2 (1-\phi^2)^3/(4r^4) \right\} dr \quad (8)$$

Integrating over  $dr$  can be made using known expressions for the Euler-type integrals, e.g.

$$\int_0^\infty \frac{dr}{1 + (r/r_0)^b} = \frac{\pi r_0}{b \sin(\pi/b)}, \quad b > 1,$$

and, more generally

$$\int_0^\infty \frac{(r/r_0)^c dr}{\beta + (r/r_0)^b} = \beta^{(1+c-b)/b} \frac{\pi r_0}{b \sin[\pi(1+c)/b]}, \quad \beta > 0, b > 1 + c, c > -1. \quad (9)$$

Differentiation in  $\beta$  allows to get the integral with any power of  $1 + (r/r_0)^b$  in denominator. What we need now is ( $\phi$  is defined in (7))

$$\begin{aligned} \int (1-\phi^2) dr &= \frac{4\pi r_0}{b^2 \sin(\pi/b)}, & \int \frac{(1-\phi^2)^2}{r^2} dr &= \frac{8\pi(1-1/b^2)}{3r_0 b^2 \sin(\pi/b)}, \\ \int \frac{(1-\phi^2)^3}{r^4} dr &= \frac{32\pi(1-9/b^2)(1-9/4b^2)}{5r_0^3 b^2 \sin(3\pi/b)} = \frac{32\pi}{15r_0^3 b^2 \sin(\pi/b)} F_3(b), \end{aligned} \quad (10)$$

with  $F_3(b) = 3 \sin(\pi/b)(1-9/b^2)(1-9/4b^2)/\sin(3\pi/b)$ . Asymptotically at large  $b$  the function  $F_3(b) \rightarrow 1$ , it is really close to 1 in wide interval of argument, from  $b \sim 6$  up to  $\infty$ , the deviation does not exceed several %. However, correction due to deviation of  $F_3$  from 1 will be taken into account.

Expressions (9,10) allow to obtain the mass of multiskyrmion in simple analytical form as a function of parameters  $b$  and  $r_0$ :

$$M(B, r_0, b) = \frac{1}{3b \sin(\pi/b)} \left[ (A_N b^2 + 8B) \frac{r_0}{b} + \frac{4}{3br_0} (1 - \lambda)(Bb^2 + 2\mathcal{I})(1 - 1/b^2) + \lambda \mathcal{I} \frac{8b}{15r_0^3} F_3(b) \right]. \quad (11)$$

Since  $b \sim 2\sqrt{B}$ , or greater, see below, we can put  $F_3 \rightarrow 1$  and substitute at large enough  $B$   $\pi/b \sin(\pi/b) \rightarrow 1$ , to obtain:

$$M(B, r_0, b) \simeq \frac{1}{3\pi} \left\{ b \left[ A_N r_0 + (1 - \lambda) \frac{4B}{3r_0} + \lambda \frac{8\mathcal{I}}{15r_0^3} \right] + \frac{8}{b} \left[ B r_0 + (1 - \lambda) \frac{\mathcal{I}}{3r_0} \right] \right\}. \quad (12)$$

Now minimization over  $b$  can be done without difficulties. It provides an upper bound for the soliton mass, because we restricted ourselves with the profiles of the type (7), only. Minimizing in  $b$  is trivial, giving

$$b_{min} = \sqrt{G_N/G_D}, \quad M(B, r_0) = \frac{2}{3\pi} \sqrt{G_N G_D} \quad (13)$$

with

$$G_N = 8 \left[ B r_0 + (1 - \lambda) \frac{\mathcal{I}}{3r_0} \right], \quad G_D = A_N r_0 + (1 - \lambda) \frac{4B}{3r_0} + \lambda \mathcal{I} \frac{8}{15r_0^3}. \quad (14)$$

Minimization of the mass in  $r_0$  provides now  $r_0^{min}$  and upper bound for the mass  $M(B)$ . The extreme cases  $\lambda = 0$  (SK4) and  $\lambda = 1$  (SK6) are simple, and we present here results for both cases.

The SK4 case, considered in [9].

$$r_0^{min} \simeq \left( \frac{2}{3} \sqrt{\frac{\mathcal{I}}{A_N}} \right)^{1/2}, \quad b_0^{min} \simeq 2(\mathcal{I}/A_N)^{1/4}, \quad \frac{M}{B} < \frac{4}{3\pi} \left( \frac{2}{3} \right)^{1/2} (2 + \sqrt{A_N \mathcal{I}/B^2}) \quad (15)$$

The SK6 case.

$$r_0^{min} \simeq \left( \frac{8\mathcal{I}}{15 A_N} \right)^{1/4}, \quad b_0^{min} = 2\sqrt{B/A_N}, \quad \frac{M}{B} < \frac{8}{3\pi} \left( \frac{8A_N \mathcal{I}}{15 B^2} \right)^{1/4}. \quad (16)$$

The masses of multiskyrmions in the SK6 case at large  $B$  are smaller than those in the SK4 case, the ratio of radii is greater:  $r_0(SK6)/r_0(SK4) = (6/5)^{1/4} \simeq 1.0466$ .

For the SK4 variant it is possible to obtain more accurate estimates describing also preasymptotics in  $B$ .

$$r_0^{min} = 2 \left[ \frac{(Bb^2 + 2\mathcal{I})(1 - 1/b^2)}{3(A_N b^2 + 8B)} \right]^{1/2} \quad (17)$$

and

$$M(B, b)/B = \frac{4}{3b \sin(\pi/b)} [(b^2 + 2\mathcal{I}/B)(A_N b^2 + 8B)(1 - 1/b^2)/(3b^2 B)]^{1/2} \quad (18)$$

At large enough  $B$  when it is possible to neglect the influence of slowly varying factors  $(1 - 1/b^2)$  and  $b \sin(\pi/b)$  we obtained [9]

$$b^{min} = b_0 = 2(\mathcal{I}/A_N)^{1/4}, \quad r_0^{min} \simeq \left[ \frac{2}{3} \left( \sqrt{\frac{\mathcal{I}}{A_N}} - \frac{1}{4} \right) \right]^{1/2} \quad (19)$$

and, therefore, approximately

$$\frac{1}{3} (2 + \sqrt{\mathcal{I} A_N / B}) < \frac{M}{B} < \frac{1}{3} (2 + \sqrt{\mathcal{I} A_N / B}) \frac{4}{b_0 \sin(\pi/b_0)} \left[ \frac{2}{3} \left( 1 - \frac{1}{b_0^2} \right) \right]^{1/2}. \quad (20)$$

The lower bound in (20) is taken from (4a).

The correction to the value  $b_0$  can be found including into minimization procedure the factor  $(1 - 1/b^2)$  and variation of  $b \sin(\pi/b) \simeq \pi[1 - \pi^2/(6b^2)]$ . It provides:

$$\delta b \simeq \frac{B(\pi^2/3 - 1)(2 + \sqrt{A_N \mathcal{I}/B})^2}{16\mathcal{I}^{3/4} A_N^{1/4}}, \quad (21)$$

and the value  $b = b_0 + \delta b$  should be inserted into (18). This improves the values of  $M/B$  for  $B = 1, 2, 3, \dots$  but provides negligible effect for  $b \sim 17$  and greater, since  $\delta b \sim 1/\sqrt{B}$ . The comparison of numerical calculation result and analytical upper bound (18) is presented in **Table 1**.

For the SK6 variant from (11) we have after minimization in  $r_0$

$$(r_0^{min})^2 = \left[ \frac{8\mathcal{I}F_3(b)b^2}{5(A_N b^2 + 8B)} \right]^{1/2} \quad (22)$$

and

$$\frac{M(B, b)}{B} = \frac{4}{9b \sin(\pi/b)} \left[ \frac{8\mathcal{I}F_3(b)}{5Bb^2} (A_N b^2/B + 8)^3 \right]^{1/4} \quad (23)$$

which provides the upper bound for the soliton mass, for any  $b$ , similar to (18). Combining with (4a), we obtain approximately

$$\frac{2}{3} \left( \frac{\mathcal{I}A_N}{B^2} \right)^{1/4} < \frac{M}{B} < \frac{8}{3b_0 \sin(\pi/b_0)} \left( \frac{8A_N \mathcal{I}F_3(b)}{15B^2} \right)^{1/4}. \quad (24)$$

The correction to the value of  $b_0 = 2\sqrt{B/A_N}$  can be taken into account similar to the SK4 case,

$$\delta b(SK6) \simeq \frac{3(2\pi^2 - 45/4)}{4b}. \quad (25)$$

The results for the upper bound for the masses of the SK6 multiskyrmions calculated according to (23) are given in **Table 2**, next Section.

A comment concerning the behaviour of the profile  $f$  at large  $r$  is necessary. It is well known that asymptotically at  $r \rightarrow \infty$   $f$  is defined by the  $2 - d$  order term in the lagrangian and is proportional to  $f \sim 1/r^p$  with  $p = 1/2 + \sqrt{2B + 1/4}$ . So,  $p = 2$  for  $B = 1$ ,  $p = 3$  for  $B = 3$ , etc. Obviously, the tail of the profile we obtained here,  $f \sim 1/r^{b/2}$  with  $b$  given in (15) or (16), is greater and falls down more slowly than the true one. This is because our purpose is to describe the masses of skyrmions and other global characteristics, but not the asymptotic behaviour of the profile.

### 3 Numerical results and comparison of the SK6 and SK4 variants of the model

Numerically (18) provides the upper bound for the skyrmion masses which differs from the masses of known  $RM$  multiskyrmions in the SK4 variant within  $\sim 2\%$ , beginning with  $B = 2$ , see **Table 1**. The values  $M/B|_{RM}$  are calculated numerically by means of minimization of the functional (3). Even for  $B = 1$ , where the method evidently should not work well, we obtained  $M = 1.271$  for  $b = 2.85$  instead of precise value  $M = 1.232$ . It should be noted here that analytical results of paper [9] for smallest  $B$  are slightly improved here due to better

choice of power  $b(B)$ : it is found by numerical minimization of (18), whereas in [9] we used approximate formula (21) for  $\delta b$ . For maximal values of  $B$  between 17 and 22 where the value of  $\mathcal{I}$  is calculated, the upper bound exceeds the  $RM$  value of mass by 0.5% only. We took here the ratio  $R_{I/B} = \mathcal{I}/B^2$  in the cases where this ratio is not determined yet, the same as for highest  $B$  where it is known, i.e. 1.28 for  $SU(2)$  case [3],  $B = 32$  and  $64$ , and 1.037 for  $B > 6$  in  $SU(3)$  [6].

|                      |        |        |        |        |        |        |        |        |        |        |       |
|----------------------|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|-------|
| $B$                  | 2      | 3      | 4      | 5      | 6      | 7      | 13     | 17     | 22     | 32     | 64    |
| $M/B _{RM}$          | 1.208  | 1.184  | 1.137  | 1.147  | 1.137  | 1.107  | 1.099  | 1.091  | 1.092  | 1.088  | 1.084 |
| $r_B$                | 1.45   | 1.73   | 1.89   | 2.13   | 2.30   | 2.40   | 3.23   | 3.65   | 4.15   | 4.97   | 6.98  |
| $b(B)$               | 3.73   | 4.38   | 4.77   | 5.35   | 5.76   | 6.00   | 7.98   | 9.01   | 10.23  | 12.24  | 17.16 |
| $M/B _{appr}$        | 1.227  | 1.198  | 1.150  | 1.158  | 1.147  | 1.117  | 1.106  | 1.0976 | 1.098  | 1.094  | 1.089 |
| $M/B _{num}$         | 1.1791 | 1.1462 | 1.1201 | 1.1172 | 1.1079 | 1.0947 | 1.0834 | 1.0774 | 1.0766 | —      | —     |
| $M/B _{RM}^{SU_3}$   | 1.222  | 1.215  | 1.184  | 1.164  | 1.145  | 1.138  | 1.120  | 1.115  | 1.111  | 1.1064 | 1.101 |
| $r_B^{SU_3}$         | 1.28   | 1.54   | 1.72   | 1.88   | 2.02   | 2.16   | 2.87   | 3.25   | 3.68   | 4.40   | 6.18  |
| $b(B)^{SU_3}$        | 3.32   | 3.96   | 4.38   | 4.76   | 5.08   | 5.43   | 7.12   | 8.05   | 9.08   | 10.86  | 15.19 |
| $M/B _{appr}^{SU_3}$ | 1.247  | 1.231  | 1.198  | 1.176  | 1.156  | 1.149  | 1.127  | 1.121  | 1.116  | 1.111  | 1.106 |

**Table 1.** The skyrmion mass per unit  $B$ -number in universal units  $3\pi^2 F_\pi/e$  for the  $RM$  configurations, the SK4 variant, approximate and precise solutions.  $r_B = \sqrt{\langle r_B^2 \rangle}$  - mean square radius of baryon number distribution, or the isoscalar radius, in units  $2/F_\pi e$ . The approximate values (upper bounds) are calculated using formula (18) with the power  $b$  minimizing it. The numerical values for the  $SU(2)$  model are from the papers [3] and earlier papers. The last 4 lines show the result for the  $SU(3)$  projector ansatz [6] and approximation to this case,  $A_N = 4/3$ . Calculations of  $M/B|_{appr}^{SU_3}$  are made also with the help of (18) with the power  $b(B)^{SU_3}$  which minimizes it.

It is of interest to compare the same quantities ( $b_0$ ,  $r_0$ ,  $M/B$ ) at large baryon numbers for the SK4 variant (original Skyrme model) and the SK6 variant.

$$b_0^2(SK6)/b_0^2(SK4) = [B^2/(\mathcal{I}A_N)]^{1/2} < 1, \quad (26)$$

$$r_0^2(SK6)/r_0^2(SK4) = (6/5)^{1/2} > 1. \quad (27)$$

Asymptotically at large  $B$  the ratio of upper and lower bounds [9]

$$R_{max/min}(SK4) = \frac{M_{max}}{M_{min}}|_{SK4} = \frac{4}{\pi} \left( \frac{2}{3} \right)^{1/2} \simeq 1.0396, \quad (28)$$

i.e. the gap between upper and lower bounds is less than 4%, independently on  $B$ , the particular value of  $\mathcal{I}$  and the number of flavours  $N$ . For the SK6 variant we obtain

$$R_{max/min}(SK6) = \frac{M_{max}}{M_{min}}|_{SK6} = \frac{4}{\pi} \left( \frac{8}{15} \right)^{1/4} \simeq 1.0881, \quad (29)$$

here the gap is less than 9%, also independently on  $B$ ,  $\mathcal{I}$  and  $N$ .

Note that the ratio

$$R_{max/min}(SK6)/R_{max/min}(SK4) = r_0(SK6)/r_0(SK4) \simeq (6/5)^{1/4} \simeq 1.0466, \quad (30)$$

so, the number  $(6/5)^{1/4}$  plays a special role in comparison of both variants of the model.

The SK6 variant reveals even weaker dependence on the quantity  $\mathcal{I}$  than the SK4 variant, the power  $b_0$  does not depend on  $\mathcal{I}$  at all, cf. (15) and (16) above.

With decreasing  $\mathcal{I}$  the upper bounds decrease proportionally to the lower bounds. It should be stressed that our upper bounds for masses of the SK4 and SK6 multiskyrmions are obtained on a class of profile functions (7), and, probably, can be improved.

|                             |        |        |        |       |       |       |       |        |       |        |        |
|-----------------------------|--------|--------|--------|-------|-------|-------|-------|--------|-------|--------|--------|
| $B$                         | 2      | 3      | 4      | 5     | 7     | 13    | 17    | 22     | 32    | 64     | 128    |
| $M/B _{RM}^{SK6}$           | 0.9181 | 0.8843 | 0.8289 | 0.837 | 0.793 | 0.781 | 0.772 | 0.773  | 0.768 | 0.763  | 0.760  |
| $b(B)$                      | 4.08   | 4.66   | 5.14   | 5.55  | 6.27  | 8.00  | 8.95  | 10.01  | 11.85 | 16.39  | 22.9   |
| $M/B _{appr}^{SK6}$         | 0.951  | 0.915  | 0.857  | 0.865 | 0.817 | 0.801 | 0.791 | 0.790  | 0.784 | 0.778  | 0.774  |
| $r_{B,RM}^{SK6}$            | 1.636  | 1.910  | 2.047  | 2.293 | 2.540 | 3.351 | 3.767 | 4.269  | 5.093 | 7.125  | 10.037 |
| $M_{RM}^{SK6}/M_{RM}^{SK4}$ | 0.760  | 0.747  | 0.729  | 0.730 | 0.716 | 0.711 | 0.708 | 0.7075 | 0.706 | 0.7044 | 0.704  |
| $r_B^{SK6}/r_B^{SK4}$       | 1.128  | 1.107  | 1.082  | 1.076 | 1.056 | 1.039 | 1.033 | 1.029  | 1.025 | 1.022  | 1.019  |

**Table 2.** The skyrmion mass per unit  $B$ -number for the SK6 variant of the model ( $\lambda = 1$ ) with  $SU(2)$  flavour symmetry, in units  $3\pi^2 F_\pi/e$ , and comparison with the SK4 variant. The approximate upper bound  $M/B|_{appr}$  is calculated using formulas (23) with the power  $b(B)$  given in third line. The isoscalar radius  $r_{B,RM}$  in units  $2/F_\pi e$ . The ratios of masses and radii  $r_B$  for the SK6 and SK4 variants are presented in the last two lines.

The masses of multiskyrmions for the SK6 variant presented in **Table 2** are considerably lower than for the SK4 variant (they are between 0.76 and 0.7 of them, roughly) for accepted choice of parameters. They agree well with numerical results of recent paper [7]. For the SK6 variant the  $B = 1$  configuration can be described by profile of the type (7) with accuracy about 3%, as for the SK4 variant:  $M(1)_{appr} \simeq 0.969$  for  $b = 3.4$  in comparison with numerical value  $M(1) = 0.940$ . At higher  $B$ -numbers the upper bound for masses of the SK6 multiskyrmions is not so close to numerical values as for the SK4 variant. The difference is not smaller than 2%, but from practical point of view it is quite good agreement. In view of good quantitative agreement of analytical and numerical results the studies of basic properties of bubbles of matter made in [9] and in present paper are quite reliable.

The width (or thickness)  $W$  of the bubble shell can be estimated easily. We can define the half-width as a distance between points where  $\phi = \pm 1/2$ , then:

$$W = 4 \frac{r_0}{b_0} \ln 3. \quad (31)$$

Looking at (6) we see that maximal value of  $\phi'$  is close to  $b_0/(2r_0)$ , and this provides immediately  $W \simeq 4r_0/b_0$ , in agreement with (31). Here there is some difference between the SK4 and SK6 variants of the model, because the ratio  $r_0/b_0$  is different. For the original SK4 variant at large  $B$  the thickness  $W \simeq 2\sqrt{2/3} \ln 3$ , i.e. it is universal characteristic of all baryonic bags, and does not depend on the number of flavours  $N$  as well. For the SK6 variant  $W \simeq 2(8A_N \mathcal{I}/15B^2)^{1/4}$ , i.e. it can depend slightly on  $B$  if the ratio  $\mathcal{I}/B^2$  has such dependence, and increases also with increasing number of flavours. The radius of the bubble grows with increasing  $B$  like  $[\mathcal{I}/A_N]^{1/4}$  for both SK4 and SK6 variants, see (15),(16).

## 4 Toy model: the inclined step approximation

A natural question is to what extent the "bubble" structure is a necessary property of multiskyrmions, and what could be instead of this. What we have is a boundary condition on profile function,  $f(0) = \pi$  and  $f(\infty) = 0$ , and requirement to get the minimal value of the mass (3). In principle, the profile  $f$  could decrease according to some law different from

(7), providing another mass and  $B$ -number distribution, e.g. uniformly filled bag, and it is just the property of lagrangian (3) that the bubble structure has an advantage. A good illustration for this provides the toy model of "inclined step" type [4]. Let  $W$  be the width of the step, and  $r_0$  - the radius of the skyrmion where the profile  $f = \pi/2$ .  $f = \pi/2 - (r - r_0)\pi/W$  for  $r_0 - W/2 \leq r \leq r_0 + W/2$ . This approximation describes the usual domain wall energy [10] with accuracy  $\sim 9.5\%$ .

We can write the energy in terms of  $W, r_0$ , then minimize it with respect to both of these parameters, and find the minimal value of energy.

$$M(W, r_0) = \frac{1}{3\pi} \left[ \frac{\pi^2}{W} \left( A_N r_0^2 + (1 - \lambda)B + \lambda \frac{3\mathcal{I}}{8r_0^2} \right) + W \left( B + (1 - \lambda) \frac{3\mathcal{I}}{8r_0^2} \right) \right] \quad (32)$$

This gives

$$W_{min} = \pi \left[ \frac{A_N r_0^2 + (1 - \lambda)B + 3\lambda\mathcal{I}/(8r_0^2)}{B + (1 - \lambda)3\mathcal{I}/(8r_0^2)} \right]^{1/2} \quad (33)$$

and the mass

$$M = \frac{2}{3} \left[ (A_N r_0^2 + (1 - \lambda)B + 3\lambda\mathcal{I}/(8r_0^2)) (B + (1 - \lambda)3\mathcal{I}/(8r_0^2)) \right]^{1/2}. \quad (34)$$

The minimization of (34) over  $r_0$  should be performed now. For pure SK4 ( $\lambda = 0$ ) and SK6 variants ( $\lambda = 1$ ) it can be made trivially, and in both cases provides the same result,  $(r_0^{min})^2 = \sqrt{3\mathcal{I}/(8A_N)} \simeq 0.612\sqrt{\mathcal{I}/A_N}$ . It is close to the above result  $(r_0^{min})^2(SK4) \simeq 0.667\sqrt{\mathcal{I}/A_N}$  and  $(r_0^{min})^2(SK6) \simeq 0.73\sqrt{\mathcal{I}/A_N}$ . In dimensional units  $r_0^{min} = (6\mathcal{I}/A_N)^{1/4}/(F_\pi e)$ .

The thickness of the envelope  $W$  is slightly different for both models. For the SK4 model  $W_{min}(SK4) = \pi$  [4], i.e. it does not depend on  $B$  for any  $SU(N)$ , similar to previous result (31) which gives  $W(SK4) \simeq 1.8$  for large  $B$ , all in units  $2/(F_\pi e)$ . For the SK6 variant  $W_{min}(SK6) = \pi[3A_N\mathcal{I}/2B^2]^{1/4} > W_{min}(SK4)$  in this toy model, also similar to result obtained in previous Section, and increases with increasing  $N$ . Numerically, effect is not great since  $A_N < 2$ . For  $SU(3)$  group the factor is  $\sim [3A_N/2]^{1/4} = 2^{1/4} \simeq 1.19$ .

The energy obtained in this way is

$$M_{min}(SK4) \simeq (2B + \sqrt{3A_N\mathcal{I}/2})/3 \quad (35)$$

and

$$M_{min}(SK6) \simeq \frac{2B}{3} \left( \frac{3A_N\mathcal{I}}{2B^2} \right)^{1/4} \simeq 0.738 B \left( \frac{A_N\mathcal{I}}{B^2} \right)^{1/4}. \quad (36)$$

In difference from previous results, (35) and (36) do not give the upper bound for the skyrmion masses since some terms in expansion in  $W$  have been neglected, and for small  $B$ , indeed, the value of (35) is smaller than calculated masses of skyrmions. For  $SU(2)$  model  $A_N = 1$  and the energy  $M_{min}(SK4) = (2B + \sqrt{3\mathcal{I}/2})/3$ . The formula gives the numbers for  $B = 3, \dots, 22$  in agreement with calculation within  $RM$  approximation within 2–3% [3, 5].

More detailed analytical calculation made in present paper confirms the results of such "toy model" approximation and both reproduce the picture of  $RM$  skyrmions as a two-phase object, a spherical bubble with profile  $f = \pi$  inside and  $f = 0$  outside, and a thickness of the shell which is fixed (the SK4 model), or slightly depends on  $B$ -number and  $N$  (the SK6 model).

The surface energy density can be estimated. For the SK4 model  $\rho_M^{surf} \simeq (2B + \sqrt{\mathcal{I}A_N})/(12\pi r_0^2) \sim 3\sqrt{A_N B^2/\mathcal{I}}/8\pi$ , or in ordinary units  $\rho_M^{surf} \sim 9\pi\sqrt{A_N B^2/\mathcal{I}}F_\pi^3 e/32$ . The average

volume mass density in the shell is, in ordinary units,

$$\rho_M^{vol} \simeq \frac{3\pi}{64W}(2B + \sqrt{\mathcal{I}A_N})\sqrt{A_N/\mathcal{I}}F_\pi^4 e^2 \sim \frac{9\pi}{64W}\sqrt{A_N B^2/\mathcal{I}}F_\pi^4 e^2. \quad (37)$$

For  $SU(2)$  model at large  $B$  it is about  $\sim (0.4 - 0.6)Gev/Fm^3$  depending on the value of  $W$  discussed above, at reasonable choice of model parameters  $F_\pi = 0.186 Gev$ ,  $e = 4.12$  [4], i.e. this density is several times greater than normal density of nuclei. For the SK6 model the density of matter in the shell is about  $\sim 0.7 - 0.8$  of density for the SK4 model.

## 5 The properties of the large $B$ multiskyrmions

It is possible to calculate analytically such characteristics of multiskyrmions as mean square radii of baryon number and mass distributions, tensors of inertia, etc. Let us estimate first the average dimensions of each cell in the envelope of the bubble of matter. The average area of the cell is  $4\pi r_0^2/(2B - 2)$ , at large  $B$  it is for the SK4 model

$$S_{cell} \simeq 4\pi\sqrt{\mathcal{I}/A_N}/(3B). \quad (38a)$$

Therefore, the average radius of the cell is

$$r_{cell} \simeq [\mathcal{I}/A_N]^{1/4}/\sqrt{3B}, \quad (39)$$

i.e. it depends slightly on the ratio  $\mathcal{I}/B$  which is close to 1 at large  $B$ . For the mentioned above choice of  $SU(2)$  model parameters  $r_{cell} \sim 0.32Fm$ , about twice smaller than radius of the  $B = 2$  torus.

For the SK6 variant the dimensions of each cell are slightly greater,

$$S_{cell} \simeq 4\pi\sqrt{2/15}\sqrt{\mathcal{I}/A_N}/B. \quad (38b)$$

The radii of the baryon number and mass distributions can be calculated analytically for the profile (7) in terms of parameters  $r_0, b$ . We have

$$\langle r_B^2 \rangle (r_0, b) = \frac{2}{\pi} \int r^2 s_f^2 f' dr = \frac{b}{\pi} \int (1 - \phi^2)^{3/2} r dr = r_0^2 \frac{b^2 - 16}{b^2 \cos(2\pi/b)} \quad (40)$$

At large  $B$ ,  $\langle r_B^2 \rangle \simeq r_0^2 [1 + (2\pi^2 - 16)/b^2]$ . Evidently, when  $B \rightarrow \infty$  and  $b \rightarrow \infty$  then  $\langle r_B^2 \rangle \rightarrow r_0^2$ , as it should be expected. But for realistic values of  $B$  and  $b$  the value of  $\langle r_B^2 \rangle$  is greater than  $r_0^2$ . When  $b \rightarrow 4$ ,  $(b^2 - 16)/[b^2 \cos(2\pi/b)] \rightarrow 4/\pi$ .

Similarly we can calculate the radii of the mass density distribution.

$$M(B, b) \langle r_M^2 \rangle = \frac{4}{3} \left[ T_1(A_N b^2/4 + 2B) + (1 - \lambda)T_2(Bb^2/2 + \mathcal{I}) + \lambda \mathcal{I} b^2 T_3/4 \right] \quad (41)$$

$$T_1 = \int (1 - \phi^2) r^2 dr = \frac{12\pi r_0^3}{b^2 \sin(3\pi/b)},$$

$$T_2 = \int (1 - \phi^2)^2 dr = \frac{8\pi r_0}{3b^2 \sin(\pi/b)} (1 - 1/b^2),$$

$$T_3 = \int (1 - \phi^2)^3 / r^2 dr = \frac{32\pi}{15r_0 b^2 \sin(\pi/b)} \left(1 - \frac{1}{b^2}\right) \left(1 - \frac{1}{4b^2}\right). \quad (42)$$

Evidently, at large  $B$   $\langle r_M^2 \rangle \rightarrow r_0^2$ , and  $\langle r_M^2 \rangle \simeq \langle r_B^2 \rangle$ .

It is possible also to calculate analytically the tensors of inertia of multiskyrmion configurations within this approximation. This will be done here for the SK4 variant which is now of greater practical importance than the SK6 variant. The explicit expressions for tensors of inertia of multiskyrmions in general  $SU(2)$  case are given in [4]. They define the rotation energy of a skyrmion in the form

$$E_{rot} = \frac{1}{2}\Theta_{ab}^I \omega_a \omega_b + \frac{1}{2}\Theta_{ab}^J \Omega_a \Omega_b + \Theta_{ab}^{int} \omega_a \Omega_b, \quad (43)$$

where angular velocities of skyrmions rotations in isotopical ( $\omega_a$ ) and usual ( $\Omega_b$ ) spaces are defined in standard way in terms of corresponding collective coordinates and their time derivatives, see, e.g., [4] and references therein. The isotopical tensor of inertia

$$\Theta_{ab}^I = \int s_f^2 \left\{ (\delta_{ab} - n_a n_b) \left[ \frac{F_\pi^2}{4} + \frac{(\vec{\partial}f)^2}{e^2} \right] + \frac{s_f^2}{e^2} \partial_l n_a \partial_l n_b \right\} d^3r \quad (44)$$

The expression for the orbital tensor of inertia is much more complicated and we shall not give it here, see [4] again. However, the traces of both tensors of inertia are simpler, especially for the  $RM$  ansatz, and depend on the quantities  $\mathcal{N} = B$  and  $\mathcal{I}$  [4]:

$$\begin{aligned} \Theta_{aa}^I &= 4\pi \int s_f^2 \left\{ \frac{F_\pi^2}{2} + \frac{2}{e^2} \left( f'^2 + B \frac{s_f^2}{r^2} \right) \right\} r^2 dr, \\ \Theta_{aa}^J &= 4\pi \int s_f^2 \left\{ B \frac{F_\pi^2}{2} + \frac{2}{e^2} \left( B f'^2 + \mathcal{I} \frac{s_f^2}{r^2} \right) \right\} r^2 dr. \end{aligned} \quad (45)$$

Evidently, the inequality takes place [4]:

$$\Theta_{aa}^J - B\Theta_{aa}^I = \frac{8\pi}{e^2} (\mathcal{I} - B^2) \int s_f^4 dr > 0 \quad (46)$$

since  $\mathcal{I} > B^2$ . The interference tensor of inertia is much smaller for all cases except the cases of spherical and axial symmetry, as for  $B = 2$ , and will not be considered here.

The point is that at large  $B$  when multiskyrmion is close to spherical bubble the diagonal components of tensors of inertia can be calculated as 1/3 of corresponding traces (45), and off-diagonal tensors of inertia being close to zero. The accuracy of these statements increases with increasing baryon number.

Now we can calculate (45) in our approximation for the profile (7). It gives:

$$\begin{aligned} \Theta_{aa}^I &= \frac{4\pi}{F_\pi e^3} [4T_1 + T_2(4B + b^2)], \\ \Theta_{aa}^J &= \frac{4\pi}{F_\pi e^3} [4BT_1 + T_2(4\mathcal{I} + Bb^2)] \end{aligned} \quad (47)$$

with  $T_1, T_2$  given above. At large baryon numbers  $T_1 \simeq 4r_0^3/b$ ,  $T_2 \simeq 8r_0/3b$ , and in natural units for tensors of inertia,  $12\pi^2/F_\pi e^3$

$$\Theta_{aa}^J/3 \simeq \frac{16}{27} \sqrt{\frac{2}{3}} \sqrt{\frac{\mathcal{I}}{A_N}} (2B + \sqrt{A_N \mathcal{I}}). \quad (48)$$

So, we obtain

$$\Theta_{aa}^J/3 \simeq \frac{2}{3} M_B r_0^2, \quad (49)$$

as it should be for empty spherical bubble with its mass concentrated in its shell. The equality (49) does not hold, however, for the contributions of second order and Skyrme

terms in the lagrangian separately. For the isotopical tensor of inertia we have inequality  $\Theta_{aa}^I/3 < 2M_B r_0^2/3B$ . The isoscalar magnetic moment of the baryonic system is defined by the orbital inertia [4],  $\mu^0 \simeq JB < r_B^2 > /3\Theta_J$ , therefore we obtain from (49)

$$\frac{\mu^0}{J} = \frac{B}{2M_B}, \quad (50)$$

almost constant for large  $B$ -numbers. The consideration of other electromagnetic and weak interaction properties of multiskyrmions is behind the framework of present paper, it will be made elsewhere.

We finish this Section with a remark that the characteristics of multiskyrmions obtained here,  $< r_B >$ ,  $< r_M >$ ,  $S_{cell}$ , moments of inertia as well as thickness and the mass density of the shell of the bubble given in (31) and (37) provide a complete picture of large  $B$  multiskyrmions as quasiclassical objects formed by the chiral fields.

## 6 The role of the mass term

Consider also the influence of the chiral symmetry breaking mass term (*M.t.*) which is described by the lagrangian

$$-\mathcal{L}_M = M.t. = \tilde{m} \int r^2 (1 - \cos f) dr, \quad \cos f = \phi \quad (51)$$

$\tilde{m} = 8\mu^2/(3\pi F_\pi^2 e^2)$ ,  $\mu = m_\pi$ . For strangeness, charm, or bottom the masses  $m_K$ ,  $m_D$  or  $m_B$  can be inserted instead of  $m_\pi$ , multiplied by corresponding flavour content of the skyrmion.

Instead of the above expression (11) we obtain now

$$M_B < M(B, r_0, b) = \alpha(B, b)r_0 + (1 - \lambda)\beta(B, b)/r_0 + \lambda\gamma(B, b)/r_0^3 + m r_0^3 \quad (52)$$

with  $\alpha, \beta, \gamma$  given in (11), (12) and  $m = 2\pi\tilde{m}/(b \sin(3\pi/b))$ . It is possible to obtain in a simple form the precise minimal value of the mass for the SK4 model ( $\lambda = 0$ )

$$M(B, b) = \frac{2r_0^{min}}{3} (\sqrt{\alpha^2 + 12m\beta} + 2\alpha) \quad (53)$$

at the value of  $r_0$

$$r_0^{min}(B, b) = \left[ \frac{\sqrt{\alpha^2 + 12m\beta} - \alpha}{6m} \right]^{1/2}. \quad (54)$$

Eq-ns. (52) and (53) give the upper bounds for the mass of the skyrmion, because they are calculated for the profile (7) different from the true profile which can be obtained by explicit minimization of the energy functional (3) with the mass term included. In particular, it is well known that the tail of the profile decreases exponentially,  $\sim \exp(-\mu r)/r$  at  $r > \sqrt{2B}/\mu$  instead of the power law. However, for multiskyrmions at large  $B$  the main contribution to the mass, moments of inertia of the skyrmion, etc. is due to the shell of the bubble, and the relative contribution of the tail (i.e. the region outside of the bubble) decreases as  $1/\sqrt{B}$ , at least, being not important at large  $B$ .

When the mass  $m$  is small enough, as for the pion, the expansion in  $12m\beta/\alpha^2$  can be made, and one obtains the following reduction of the skyrmion size  $r_0$ :

$$r_0 \rightarrow r_0 - \frac{3m}{2\alpha} \left( \frac{\beta}{\alpha} \right)^{3/2} \simeq \sqrt{\frac{2}{3}} \left( \frac{\mathcal{I}}{A_N} \right)^{1/4} \left[ 1 - \frac{3\pi m}{2(2B + \sqrt{\mathcal{I}A_N})} \left( \frac{\mathcal{I}}{A_N} \right)^{3/4} \right], \quad (55)$$

and increase of the soliton mass

$$\delta M = M \frac{m\beta}{2\alpha^2} \left[ 1 - \frac{9m\beta}{8\alpha^2} \right] \simeq M m \frac{\pi(\mathcal{I}A_N)^{3/4}}{2(2B + \sqrt{\mathcal{I}A_N})}. \quad (56)$$

We used that at large  $B$

$$\alpha \simeq \frac{1}{3\pi} \left( A_N b + \frac{8B}{b} \right) \sim \sqrt{B}, \quad \beta \simeq \frac{4}{9\pi} \left( Bb + \frac{2\mathcal{I}}{b} \right) \sim B^{3/2}, \quad \gamma = \frac{8}{45\pi} \mathcal{I} \sim B^2. \quad (57)$$

As it was expected from general grounds, dimensions of the soliton decrease with increasing  $m$ . However, even for large value of  $m$  the structure of multiskyrmion at large  $B$  remains the same: the chiral symmetry broken phase inside the spherical wall where the main contribution to the mass and topological charge is concentrated [4]. The value of the mass density inside of the bubble is defined completely by the mass term with  $1 - \phi = 2$ . The baryon number density distribution is quite similar, with only difference that inside the bag it equals to zero. Eq. (52),(53) provide the upper bound for the skyrmion mass, the lower bound (4a) takes place again for the mass without the mass term. It follows from these results that  $RM$  approximated multiskyrmions cannot model real nuclei at large  $B$ , probably  $B > 12 - 20$ , and configurations like skyrmion crystals [11] may be more valid for this purpose.

One of the issues of multiskyrmions phenomenology are the multibaryons with flavour different from that of  $u, d$  quarks, in particular  $s, c$  and  $b$  flavours, see [12, 4] and references therein. The important ingredient of the calculation of spectra of such multibaryons is calculation of the flavour excitation energies  $\omega_{B,s}$ ,  $\omega_{B,c}$  and  $\omega_{B,b}$ . The behaviour of these energies as a function of the baryon number is important for the conclusion if the corresponding flavour is more bound when  $B$  increases, or less bound. The following expression for these energies was obtained ([12, 4] and references therein):

$$\omega_{B,F} = \frac{N_c B}{8\Theta_{F,B}} (\mu_{F,B} - 1) \quad (58)$$

with

$$\mu_{F,B} \simeq [1 + 16\bar{m}_D^2 \Gamma_B \Theta_{F,B} / (N_c B)^2]^{1/2} \quad (59)$$

and the  $\sigma$ -term  $\Gamma_B$  and "flavour" moment of inertia  $\Theta_{F,B}$  given by

$$\Gamma_B = \frac{F_\pi^2}{2} \int (1 - c_f) d^3 r, \quad \Theta_{F,B} = \frac{1}{8} \int (1 - c_f) \left[ F_D^2 + \frac{1}{e^2} ((\vec{\partial}f)^2 + s_f^2 (\vec{\partial}n_i)^2) \right], \quad (60)$$

$N_c$  is the number of colours of the underlying QCD, the last term in (58), proportional to  $N_c$ , is due to Wess-Zumino term present in the action of the model. We neglected the terms of relative order  $(F_D^2 - F_\pi^2)/m_D^2$  because, as it was shown explicitly in [12], they are important only for small values of  $B$ , and make vanishing contribution for the large  $B$  which we consider here. The terms  $\sim (1 - c_f)$  in the integrand of (60) give contribution proportional to the volume of skyrmion,  $\sim r_0^3$ , whereas terms  $\sim s_f^2$  or  $(\partial F)^2$ , etc, are due to the shell of the skyrmion, only,  $\sim r_0^2$ .  $\bar{m}_D^2 = F_D^2 m_D^2 / F_\pi^2 - m_\pi^2$ ,  $m_D$  is the mass of flavoured meson ( $K, D$  or  $B$ -meson). When  $\bar{m}_D$  is sufficiently large - practically it is always fulfilled - then

$$\omega_{F,B} \simeq \frac{\bar{m}_D}{2} \left( \frac{\Gamma_B}{\Theta_{F,B}} \right)^{1/2} - \frac{N_c B}{8\Theta_{F,B}} \quad (61)$$

For small  $B$  the energies decrease somewhat with increasing  $B$ , and this leads to the increase of binding energies of flavours. From the picture of large  $B$  multiskyrmions clarified here we can see that the quantity  $\Gamma_B$  has the part proportional to the volume occupied by multiskyrmion, i.e.  $\sim B^{3/2}$ , whereas for  $\Theta_{F,B}$  this part is considerably smaller. As a result, at large  $B$  the ratio

$$\frac{\Gamma_B}{\Theta_{B,F}} \rightarrow 4 \frac{F_\pi^2}{F_D^2} = \frac{4}{\rho_D^2} \quad (62)$$

For the difference  $\epsilon_{F,B} = m_D - \omega_{F,B}$  which is important contribution into the binding energy of flavoured meson by  $SU(2)$  skyrmion we obtain

$$\epsilon_{F,B} \simeq \frac{1}{2} \frac{m_\pi^2}{m_D \rho_D^2} + 2m_D \frac{\Theta_{F,B}^{surf}}{\rho_D^2 \Gamma_B} + \frac{N_c B}{2\Gamma_B \rho_D^2}, \quad (63)$$

where the surface, or shell, contribution to flavour inertia  $\Theta_{F,B}^{surf}$  is proportional to  $1/e^2$  in (60). Only the first term in (63) comes from the volume of the multiskyrmion, but numerically it is negligible, except the case of the strangeness. Remaining terms are both of the surface (or shell) origin, in other words, the binding of heavy flavour takes place on the shell of the multiskyrmion. The second term,  $\sim \Theta_{F,B}^{surf}/\Gamma_B$ , which dominates in magnitude, decreases with increasing  $B$  as  $1/\sqrt{B}$ , and this explains the results of calculations [4, 12] which have shown that the binding of flavour becomes weaker with increasing  $B$ . However, this property may be intrinsic for  $RM$  multiskyrmions, and can be absent for the skyrmion crystals, for example.

## 7 Discussion and conclusions

In [9] and in the present paper we established the link between the topological soliton models in rational maps approximation, for two different modifications of the Skyrme model, the SK4 and SK6, and the soliton models of "domain wall" type. It was possible by means of consideration of a class of simple functions (7) approximating both the skyrmion profiles and the spherical domain walls. The ansatz (7) does not describe correctly the asymptotical behaviour of the profile at large distances, although provides quite accurate description of many characteristics of multiskyrmions. This simple picture of the  $RM$  multiskyrmions allows to understand some peculiarities of multiskyrmions phenomenology which appeared as a result of the calculations, or as computer evidence.

The upper bound for the energy of multiskyrmions is obtained which is very close to the known energies of the  $RM$  multiskyrmions, especially at largest  $B$ , and is higher than the known lower bound by  $\sim 4\%$  only for the SK4 (pure Skyrme) variant of the model, and by  $\sim 9\%$  for the SK6 variant. For the SK6 variant the upper bound obtained is considerably lower than for the SK4 variant, dimensions of solitons are larger by few %, at asymptotically large baryon numbers for accepted choice of model parameters.

The following properties of bubbles of matter from  $RM$  multiskyrmions are established analytically, mostly independent on particular values of the quantity  $\mathcal{I}$ :

The dimensions of the bubble grow with  $B$  as  $\sqrt{B}$ , or as  $\mathcal{I}^{1/4}$ , whereas the mass is proportional to  $\sim B$  at large  $B$ . Dimensions of the bubble decrease slightly with increasing  $N$  - the number of flavours,  $r_0 \sim [N/(2(N-1))]^{1/4}$ , see (15), (16).

For the SK4 variant the thickness of the bubbles envelope (31) is constant at large  $B$  and does not depend on the number of flavours, therefore, the average surface mass density

is constant at large  $B$ , as well as average volume density of the shell (37). Both densities increase slightly with increasing  $N$ . For the SK6 variant the thickness of the shell slightly depends on the ratio  $\mathcal{I}/B^2$ . At the same time the mass and  $B$ -number densities of the whole bubble  $\rightarrow 0$  when  $B \rightarrow \infty$ , and this is in contradiction with nuclear physics data confirming the constant density of nuclear matter. The bubble structure of multiskyrmions is more pronounced for the SK4 variant of the model. The material which the shell of the bag is made of looks like honeycomb, or web with constant average area of each cell.

The treatment performed in the present paper could be generalized in several directions. More general effective lagrangians can be considered, including higher terms in chiral derivatives, and the bundle of profiles can be introduced instead of one, see (7):

$$\phi = \int \rho(b) \frac{(r/r_0)^b - 1}{(r/r_0)^b + 1} db \quad (64)$$

with the evident normalization condition  $\int \rho(b) db = 1$ . The multiskyrmion mass can be written then in the form:

$$M(B) = \mu_2(B)r_0 + \mu_4(B)/r_0 + \mu_6(B)/r_0^3 + \mu_8(B)/r_0^5 + \dots \quad (65)$$

with  $\mu_2(B) = \int \rho(b)\alpha(B, b)db$ , etc. It would be of interest to study the advantages and prospects of such more general approach. Some models for higher order terms in the lagrangian have been considered in [13] for the  $SU(2)$  skyrmions.

It follows from the above consideration that the spherical bubble or bag configuration can be obtained from the lagrangian written in terms of chiral degrees of freedom only, i.e. the Skyrme model lagrangian leads at large baryon numbers to the formation of spherical bubbles of matter and thus provides a field-theoretical realization of the bag-type model. In such models, which have been popular a time ago, the properties of the building material of bags have been postulated [14]. It was the technical obstacle in formulating the bag model as a consistent quantum theory: the bag boundary  $R(t)$  was prescribed externally, and special efforts have been made to overcome this [15], not successful completely. The chiral soliton models leading to existence of multiskyrmions are free of this drawback. It should be noted that there is difference of principle between hadronic bags of the MIT type and the bubbles of matter which appear in effective field theories as  $RM$  multiskyrmions. In the first case the bag is the region where the quark-gluon phase of matter is restricted, in the latter case it is the region with skyrmion profile function  $f = \pi$ , different from its vacuum value  $f = 0$ . The quark-gluon phase does not enter the consideration at all, although it can exist inside the skyrmionic bubble, due to the known Cheshire Cat principle [16, 17].

This picture of the mass and  $B$ -number distribution in the  $RM$  multiskyrmions contradicts to what is known about nuclei, however, it emphasises the role of periphery of the nucleus and could be an argument in favour of shell-type models of nuclei. The skyrmion crystals [11] are believed to be more adequate for modelling nuclear matter.

It would be of interest to perform the investigation of the dynamics of bubbles in the chiral soliton models similar to that performed recently for the simplified two-component sigma model, or the sine-Gordon model in  $(3+1)$  dimensions [18]. Observations concerning the structure of large  $B$  multiskyrmions made here can be useful in view of possible cosmological applications of Skyrme-type models, see e.g. [19]. The large scale structure of the mass distribution in the Universe [20] is similar to that in topological soliton models, and

it can be the consequence of the similarity of the laws in micro- and macroworld. We conclude with a remark that analytical methods which are not typical for studies of skyrmion properties (see also [21]) allow to obtain very simple and transparent results which accuracy increases with increasing baryon number.

As it was noted by referee, the parametrization of the profile function similar to (7) was proposed long ago in [22] for  $B = 1$  hedgehog. It was  $f(r, r_0) = 2 \operatorname{atan}(r_0/r)^2$ , or

$$\cos f = \frac{(r/r_0)^4 - 1}{(r/r_0)^4 + 1},$$

which has correct asymptotics at large  $r$  with the appropriate choice of  $r_0$ , but does not minimize the mass of the skyrmion. Parametrization (7) with the power  $b$  as a parameter to be fitted provides description of masses especially good for the large baryon numbers.

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